

Here, we overview and discuss some of the most important results related to projections in von Neumann algebras.

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Comparison of Projections

Recall that if H is a Hilbert space, an element $w \in B(H)$ is called a partial isometry if, for any $h \in \ker(w)^\perp$, we have $\|Wh\| = \|h\|$. We call $\ker(w)^\perp$ the initial space of W and $\text{im}(w)$ the final space of W .

There are a variety of equivalent definitions for partial isometries.

Proposition: If $w \in B(H)$, then the following are equivalent:

- (i) w is a partial isometry;
- (ii) w^* is a partial isometry;
- (iii) w^*w is a projection onto the initial space of w ;
- (iv) ww^* is a projection onto the final space of w ;
- (v) $ww^*w = w$;
- (vi) $w^*ww^* = w^*$.

Theorem (Polar Decomposition): Let $a \in B(H)$. Then, there is a partial isometry $w \in B(H)$ with initial space $\ker(a)^\perp$ and final space $\overline{\text{im}(a)}$ such that $a = w|a|$.

If $a \in M \subseteq B(H)$, where M is a von Neumann algebra, then both $|a|$ and w are in M .

Equivalence of Projections

If $M \subseteq B(H)$ is a von Neumann algebra, then we say two projections $p, q \in P(M)$, where $P(M)$ denotes the space of projections of M , are (Murray–von Neumann) *equivalent* in M if there is a partial isometry $v \in P(M)$ such that $v^*v = p$ and $vv^* = q$. We will write $p \sim q$.

Note that projections have an ordering by saying that $p \leq q$ if $pq = qp = p$, or $\text{im}(p) \subseteq \text{im}(q)$. This allows us to say that p is *sub-equivalent* to q (in M), written $p \leq q$, if there is a partial isometry $v \in M$ such that $v^*v = p$ and $vv^* \leq q$.¹

The sub-equivalence relation in fact forms a partial order, and equivalence as projections forms an equivalence relation. We will first show that it is a preorder.

Proposition: In a von Neumann algebra, the relation $\sim$ is an equivalence relation on $P(M)$, and the relation $\leq$ is a preorder.

¹We will say that the projection q majorizes p if $p \leq q$, and we will say that q dominates p if $p \leq q$.

Proof. Reflexivity follows from the fact that projections are partial isometries, and symmetry follows from the fact that if v is a partial isometry, then so is v^* .

Now, we will show transitivity for $\leq$, from which we will see that $\sim$ is transitive. Letting $p, q, r \in P(M)$ be such that $p \leq q$ and $q \leq r$, we have partial isometries $u, v \in M$ with $u^*u = p$, $uu^* \leq q$, $v^*v = q$, and $vv^* \leq r$. Then, we have

$$\begin{aligned} qu &= qu u^* u \\ &= (qu u^*) u \\ &= u u^* u \\ &= u, \end{aligned}$$

so that

$$\begin{aligned} (vu)^*(vu) &= u^* v^* v u \\ &= u^* q u \\ &= u^* u \\ &= p \\ (vu)(vu)^* &= v u u^* v^* \\ &\leq v q v^* \\ &= v v^* v v^* \\ &= v v^* \\ &\leq r. \end{aligned}$$

Therefore, $p \leq r$, so $\leq$ is a transitive relation. $\square$

To see that $\leq$ is a partial order, we need an analogue of the Cantor–Schröder–Bernstein theorem for projections. In fact, it can be proven in a similar manner. First, we discuss a simple lemma.

Lemma: Let $M \subseteq B(H)$ be a von Neumann algebra. If $\{p_i\}_{i \in I}$ and $\{q_i\}_{i \in I}$ are pairwise orthogonal families of projections with $p_i \leq q_i$, then $\sum_{i \in I} p_i \leq \sum_{i \in I} q_i$.

Proof. Let u_i be the partial isometries with $u_i^* u_i = p_i$ and $r_i := u_i u_i^* \leq q_i$. Then, the r_i are pairwise orthogonal since the q_i are pairwise orthogonal, and for any $i \neq j$,

$$\begin{aligned} u_i^* u_j &= u_i^* u_i u_i^* u_j u_j^* u_j \\ &= u_i r_i r_j u_j \\ &= 0 \\ u_i u_j^* &= u_i u_i^* u_i u_j^* u_j u_j^* \\ &= u_i p_i p_j u_j^* \\ &= 0. \end{aligned}$$

Consequently, we get

$$\begin{aligned} \left(\sum_{i \in I} u_i^* \right) \left(\sum_{j \in I} u_j \right) &= \sum_{i \in I} u_i^* u_i \\ &= \sum_{i \in I} p_i \\ \left(\sum_{i \in I} u_i \right) \left(\sum_{j \in I} u_j^* \right) &= \sum_{i \in I} u_i u_i^* \end{aligned}$$

$$\leq \sum_{i \in I} q_i.$$

This gives $\sum_{i \in I} p_i \leq \sum_{i \in I} q_i$. □

Theorem: If $e \leq f$ and $f \leq e$, then $e \sim f$.

Proof. We will let $e_0 := e$ and $f_0 := f$. Let v and w be partial isometries with $v^*v = e$, $vv^* = f_1 \leq f$, $w^*w = f$, $ww^* = e_1 \leq e$. Inductively define a sequence of projections as follows.

Since v maps the range of e_1 isometrically onto the range of some projection dominated by f_1 , it follows that we may write $f_2 := ve_1(v e_1)^*$ with $f_2 \leq f_1$. Since w maps the range of f_1 onto the range of some projection dominated by e_1 , it follows that we may write $wf_1(w f_1)^* =: e_2$. Observe also that $v(e - e_1)$ is a partial isometry with initial projection $e - e_1$ and final projection $f_1 - f_2$.

Inductively, we obtain decreasing sequences of projections $(e_n)_n$ and $(f_n)_n$ where v maps the range of e_n isometrically onto that of f_{n+1} , and w maps the range of f_n isometrically onto that of e_{n+1} . Defining $e_\infty := \inf_n e_n$ and $f_\infty := \inf_n f_n$, we have that v maps the range of e_∞ onto that of f_∞ , and w that of f_∞ onto the range of e_∞ . Note that we have $e_\infty \sim f_\infty$.

As discussed earlier, we have that $e_n - e_{n+1} \sim f_{n+1} - f_{n+2}$, so since sums of pairwise orthogonal families of projections respects equivalence, we have

$$\begin{aligned} \sum_{n=0}^{\infty} (e_{2n} - e_{2n+1}) &\sim \sum_{n=0}^{\infty} (f_{2n+1} - f_{2n+2}) \\ \sum_{n=0}^{\infty} (e_{2n+1} - e_{2n+2}) &\sim \sum_{n=0}^{\infty} (f_{2n} - f_{2n+1}). \end{aligned}$$

Therefore, we get

$$\begin{aligned} e &= e_\infty + \sum_{n=0}^{\infty} (e_{2n} - e_{2n+1}) + \sum_{n=0}^{\infty} (e_{2n+1} - e_{2n+2}) \\ &\sim f_\infty + \sum_{n=0}^{\infty} (f_{2n+1} - f_{2n+2}) + \sum_{n=0}^{\infty} (f_{2n} - f_{2n+1}) \\ &= f. \end{aligned}$$

□

Central Projections and the Comparison Theorem

The projections in a von Neumann algebra form a complete lattice, as the collection of closed subspaces of H form a complete lattice under the operations

$$\begin{aligned} \bigvee_{i \in I} X_i &:= \overline{\sum_{i \in I} X_i} \\ \bigwedge_{i \in I} X_i &:= \bigcap_{i \in I} X_i. \end{aligned}$$

If $S \subseteq H$ is any subset, then we will define the range projection of S by

$$[S] := P_{\overline{\text{span}(S)}}.$$

Proposition: If $M \subseteq B(H)$ is a von Neumann algebra, and $x \in M$, then $[xH]$ and $[x^*H]$ are in M , with $[xH] \sim [x^*H]$ in M .

Proof. Let $x = v|x|$ be the polar decomposition. Note that $v \in M$. Now, vv^* is the projection onto $\overline{xH}$ and v^*v is the projection onto $\ker(x)^\perp = \overline{x^*H}$. Thus, these projections are equivalent in M . □

Definition: Let $x \in M$. We define the *central support* to be the projection

$$z(x) = \inf\{w \in P(Z(M)) \mid xw = wx = x\}.$$

We say p and q are centrally orthogonal if $z(p)z(q) = 0$.

Lemma: If $M \subseteq B(H)$ is a von Neumann algebra, then the central support of any $p \in P(M)$ is given by

$$z(p) = [MpH].$$

Let $w = [MpH]$. Since M is unital, it follows that $p \leq w$, and since $\overline{MpH}$ is a reducing subspace for both M and M' , we have $w \in M \cap M'$, so $z(p) \leq w$.

Conversely, if $x \in M$, then

$$\begin{aligned} xpH &= xz(p)pH \\ &= z(p)xpH, \end{aligned}$$

meaning that $[xpH] \leq z(p)$, so $w \leq z(p)$ as x was arbitrary.

Proposition: Let M be a von Neumann algebra, and let $p, q \in P(M)$ be projections. The following are equivalent:

- (i) p and q are centrally orthogonal;
- (ii) $pMq = \{0\}$;
- (iii) there do not exist projections $0 < p_0 \leq p$ and $0 < q_0 \leq q$ with $p_0 \sim q_0$.

Proof. Let p and q be centrally orthogonal. Then, for any $x \in M$, we have

$$\begin{aligned} pxq &= pz(p)xz(q)q \\ &= pxz(p)z(q)q \\ &= 0. \end{aligned}$$

Therefore, $pMq = \{0\}$. Now, if $pMq = \{0\}$, then $pz(q) = [MqH] = 0$, so $p \leq 1 - z(q)$. Since $1 - z(q) \in Z(M)$, we have $z(p) \leq 1 - z(q)$, meaning that $z(p)z(q) = 0$. Therefore, (i) and (ii) are equivalent.

Suppose (ii) is not the case. Let $x \in M$ be such that $pxq \neq 0$. Then, $qx^*p \neq 0$. Defining

$$\begin{aligned} p_0 &= [pxqH] \\ q_0 &= [qx^*pH], \end{aligned}$$

we have that $p_0 \leq p$, $q_0 \leq q$, and since $(pxq)^* = qx^*p$, we have $p_0 \sim q_0$.

Now, if there are nonzero projections $p_0 \leq p$ and $q_0 \leq q$ such that $p_0 \sim q_0$, then if v is a partial isometry with $v^*v = p_0$, $vv^* = q_0$, then $v^* = p_0v^*q_0$, meaning

$$\begin{aligned} pv^*q &= pp_0v^*q_0q \\ &= p_0v^*q_0 \\ &= v^* \\ &\neq 0, \end{aligned}$$

meaning $pMq \neq \{0\}$. □

Theorem (Comparison Theorem): Let $M \subseteq B(H)$ be a von Neumann algebra. For any $p, q \in P(M)$, there is a central projection $z \in P(Z(M))$ such that $pz \leq qz$ and $q(1 - z) \leq p(1 - z)$.

Proof. By Zorn's Lemma, there exist maximal families $\{p_i\}_{i \in I}$ and $\{q_i\}_{i \in I}$ of pairwise orthogonal projections

with $p_i \sim q_i$ and, setting

$$p_0 = \sum_{i \in I} p_i$$

$$q_0 = \sum_{i \in I} q_i,$$

we have $p_0 \leq q_0$. From above, we have that $p_0 \sim q_0$.

Let $w := z(q - q_0)$. Since $\{p_i\}_{i \in I}$ and $\{q_i\}_{i \in I}$ are maximal, it follows that $z(q - q_0)$ and $z(p - p_0)$ are centrally orthogonal, yielding $(p - p_0)w = 0$, meaning $pw = p_0w$.

If we let v be a partial isometry implementing the equivalence $p_0 \sim q_0$, then we have that vw is a partial isometry implementing the equivalence $p_0w \sim q_0w$. Therefore, we have

$$pw = p_0w$$

$$\sim q_0w$$

$$\leq q.$$

Similarly, $p_0(1 - w) \sim q_0(1 - w)$, so since $q - q_0 \leq w$, we have $q(1 - w) \leq p(1 - w)$. □

Recall that a factor is a von Neumann algebra M such that $Z(M) = \mathbb{C}1$.

Corollary: If M is a factor, then any two projections in M can be compared.

The Type Decomposition

Definition: Let M be a von Neumann algebra, and $p \in B(H)$ a projection not necessarily in M . The algebra pMp is known as a corner (or compression) of M .

Theorem: Let $M \subseteq B(H)$ be a von Neumann algebra, and let $p \in P(M)$. Then, pMp and $M'p$ are von Neumann algebras in $B(pH)$, and $(pMp)' = M'p$, $(M'p)' = pMp$.

Corollary: If M is a factor and $p \in P(M)$, then pMp and $M'p$ are both factors.

Definition: Let $M \subseteq B(H)$ be a von Neumann algebra. We say a projection $p \in P(M)$ is

- minimal if $p \neq 0$ and the only subprojections of p are 0 and p ;
- abelian if pMp is abelian;
- finite if $q \leq p$ and $q \sim p$ implies $q = p$;
- semifinite if there are pairwise orthogonal finite projections $p_i \in P(M)$ such that $p = \sum_{i \in I} p_i$;
- purely infinite if $p \neq 0$ and there is no nonzero finite projection $q \leq p$;
- properly infinite if $p \neq 0$ and zp is not finite for any nonzero central projection $z \in Z(P(M))$.

We say that the von Neumann algebra M is finite/semifinite/purely infinite/properly infinite if the projection 1 satisfies its respective condition. Additionally, if M has no minimal projections, we say it is diffuse.

What we will be working towards is known as the type decomposition, which forms the basis for the dimension theory of von Neumann algebras. Eventually, we will show that every von Neumann algebra M can be decomposed as

$$M = M_{\text{sf}} \oplus M_{\text{III}},$$

where M_{sf} is a semifinite von Neumann algebra and M_{III} is a type III von Neumann algebra, a definition we will discuss shortly. First, we must expand on some of the ways that comparison of projections interacts with these properties of those projections.

Lemma: Let $\{p_i\}_{i \in I}$ be a family of centrally orthogonal projections in a von Neumann algebra $M \subseteq B(H)$. If each p_i is abelian (finite), then the sum $\sum_{i \in I} p_i$ is also abelian (finite).

Proof. If each p_i is abelian, then since they are centrally orthogonal for any $i \neq j$, then for any $x, y \in M$, we have $p_i x p_i y p_i = 0$. Therefore, we have

$$\begin{aligned} (p x p)(p y p) &= \sum_{i \in I} p_i x p_i y p_i \\ &= (p y p)(p x p), \end{aligned}$$

so p is abelian.

Now, if each p_i is finite, and $u \in M$ is such that $u u^* \leq u^* u = p$, then for all i we have $z(p_i) u^* u z(p_i) = p$, and $u z(p_i) u^* = z(p_i) u u^* \leq p_i$, meaning $u z(p_i) u^* = p_i$, and

$$\begin{aligned} u u^* &= u z(p) u^* \\ &= \sum_{i \in I} u z(p_i) u^* \\ &= p. \end{aligned}$$

□

Proposition: Let p, q be nonzero projections in a von Neumann algebra with $p \leq q$. If q is finite (purely infinite), then p is also finite (purely infinite).

Proof. Suppose q is finite, and $p \sim q$, with $v \in M$ implementing the equivalence. Let $u \in M$ be such that $u^* u = p$ and $u u^* \leq p$. Then,

$$\begin{aligned} (v u v^*)^* (v u v^*) &= q \\ (v u v^*) (v u v^*)^* &\leq q, \end{aligned}$$

so since q is finite, $(v u v^*) (v u v^*)^* = q$, so $u u^* = p$.

Now, if $p \leq q$, then if $u^* u = p$ with $u u^* \leq p$, then by setting $w = u + (q - p)$, we have that $w^* w = q$ and $w w^* \leq q$, so $u^* u + (q - p) = w w^* = q$, meaning $u u^* = p$.

In the general case, we have some $q_0 \leq q$ such that $p \sim q_0 \leq q$.

Since projections are purely infinite when they have no nonzero finite subprojections, the purely infinite case follows from the finite case. □

Proposition: A projection $p \in P(M)$ is semifinite if and only if p is the supremum of finite projections. In particular, a supremum of semifinite projections is also semifinite.

Proof. If p is semifinite, then p is the sum (and hence the supremum) of a family of pairwise orthogonal finite projections.

Now, suppose $p = \bigvee_{\alpha} p_{\alpha}$, where each p_{α} is finite. Let $\{q_{\beta}\}_{\beta}$ be a maximal family of pairwise orthogonal finite subprojections of p . If we set

$$q_0 = p - \sum_{\beta} q_{\beta},$$

and suppose that $q_0 \neq 0$, then there exists some p_{α} such that p_{α} and q_0 are not orthogonal (else it would contradict maximality), so they are not centrally orthogonal. Therefore, we have a nonzero subprojection $q_1 \leq q_0$ such that $q_1 \leq p_{\alpha}$, so it is finite by what we showed previously; this contradicts the maximality of the set $\{q_{\beta}\}_{\beta}$. □

Corollary: Let p be a projection in a von Neumann algebra M . If p is semifinite (purely infinite), then the central support $z(p)$ is also semifinite (purely infinite).

Proof. The central support is the supremum over all equivalent projections, so since the supremum of semifinite projections is again semifinite, it follows from the previous proposition. $\square$

Furthermore, a nonzero projection is purely infinite if and only if it is centrally orthogonal to every semifinite projection, so we obtain the corollary in this case. $\square$

Corollary: Let p, q be nonzero projections in a von Neumann algebra such that $p \leq q$. If q is semifinite, then so is p .

Proof. It is enough to consider the case when q is central, in which case we may take $p \leq q$. Let p_0 be the maximal semifinite subprojection of p . Since q is semifinite, it is the supremum of its finite subprojections. Since $z(p - p_0) \leq q = z(q)$, it follows that if $p - p_0 \neq 0$, then there would exist a nonzero finite subprojection that would be equivalent to a subprojection of $p - p_0$, which contradicts the definition of p_0 . Thus, p is the supremum of its finite subprojections, so it is semifinite. $\square$

Lemma: Let M be a properly infinite von Neumann algebra. Then, there exists a projection $p \in P(M)$ such that $p \sim 1 - p \sim 1$.

Proof. Since M is properly infinite, there exists $u \in M$ with $uu^* \leq u^*u = 1$. Set $p_0 = 1 - uu^*$. Then, $p_n = u^n p_0 (u^n)^*$ is a pairwise orthogonal family of equivalent projections. Let $\{q_i\}_{i \in I}$ be a maximal family of pairwise orthogonal equivalent projections in M extending $\{p_n\}_{n \in \mathbb{N}}$, and set $q_0 = 1 - \sum_{i \in I} q_i$.

By the comparison theorem, there is $z \in Z(P(M))$ such that $q_0 z \leq q_{i_0} z$ and $q_{i_0} (1 - z) \leq q_0 (1 - z)$. If it were the case that $z = 0$, then we would have $q_{i_0} \leq q_0$, contradicting maximality of $\{q_i\}_{i \in I}$, so $z \neq 0$, and we have

$$\begin{aligned} z &= q_0 z + \sum_{i \in I} q_i z \\ &\leq q_{i_0} z + \sum_{i \neq i_0} q_i z \\ &= \sum_{i \in I} q_i z \leq z, \end{aligned}$$

so $z \sim \sum_{i \in I} q_i z$ by Cantor–Schröder–Bernstein for projections. Decomposing $\{q_i\}_{i \in I}$ into infinite subsets, we may construct two projections p and $z - p$ such that $p \sim z - p \sim z$.

Now, let $\{r_j\}_{j \in J}$ be a maximal family of centrally orthogonal projections with $r_j \sim z(r_j) - r_j \sim z(r_j)$. Then, the argument above shows that $\sum_{j \in J} z(r_j) = 1$, so by setting $p = \sum_{j \in J} r_j$, we obtain our desired result. $\square$

Proposition: Let p and q be finite projections in a von Neumann algebra M . Then, $p \vee q$ is finite.

Proof. We use Kaplansky's formula for this, which gives $p \vee q - p \sim q - p \wedge q$. This follows from observing that $x = (1 - p)q$ has $\ker(x) = \ker(q) \oplus (qH \cap pH)$, meaning that $[x^*H] = 1 - ((1 - q) + q \wedge p) = q - q \wedge p$. Symmetrically, this gives

$$\begin{aligned} [xH] &= (1 - p) - (1 - p) \wedge (1 - q) \\ &= p \vee q - p. \end{aligned}$$

In particular, since $q - p \wedge q \leq q$, we may assume that p and q are orthogonal, replacing q with $p \vee q - p$. We may also assume that $p + q = 1$ by passing to $(p + q)M(p + q)$.

Let z_0 be the supremum of all finite central projections. It follows that z_0 is finite. If $z_0 = 1$, then we are done. Else, we may use $(1 - z_0)p$ and $(1 - z_0)q$, wherein $z_0 = 0$ and thus we assume that M is properly infinite.

Therefore, we have a projection $r \in P(M)$ such that $r \sim 1 - r \sim 1$. By comparison, there is $z \in P(Z(M))$ such that

$$\begin{aligned} z(p \wedge r) &\leq z(q \wedge (1 - r)) \\ (1 - z)(q \wedge (1 - r)) &\leq (1 - z)(p \wedge r). \end{aligned}$$

Additionally, $zr \sim z(1-r) \sim z$, and

$$\begin{aligned} z(p \wedge r) &= zp \wedge zr \\ &\leq z(1-r) \wedge zq, \end{aligned}$$

so by using Kaplansky's formula, we have

$$\begin{aligned} zr &= z(r - r \wedge p) + z(r \wedge p) \\ &\leq z(r \vee p - p) + z(q \wedge (1-r)) \\ &= zq, \end{aligned}$$

Therefore, $zr = 0$ since zq is finite and M is properly infinite, so $z \sim 0$. In particular, this gives $q \wedge (1-r) \leq p \wedge r$. Replacing p with q and r with $1-r$, we get

$$\begin{aligned} 1-r &= (1-r - (1-r) \wedge q) + ((1-r) \wedge q) \\ &\leq ((1-r) \vee q - q) + (p \wedge r) \\ &= p, \end{aligned}$$

which gives a contradiction since p is finite. □

Proposition: Let p and q be finite projections with $p \sim q$. Then, $1-p$ and $1-q$ are equivalent.

Proof. We have that $p \vee q$ is finite, so we may assume that M is finite. By the comparison theorem, there are projections p_1 and q_1 and a central projection $z \in P(Z(M))$ such that

$$\begin{aligned} (1-p)z &\sim q_1 \leq (1-q)z \\ (1-q)(1-z) &\sim p_1 \leq (1-p)(1-z). \end{aligned}$$

Then, we have

$$\begin{aligned} z &= (1-p)z + pz \\ &\sim q_1 + qz \\ &\leq (1-q)z + qz \\ &= z \end{aligned}$$

and

$$\begin{aligned} (1-z) &= (1-q)(1-z) + q(1-z) \\ &\sim p_1 + p(1-z) \\ &\leq (1-p)(1-z), \end{aligned}$$

so since both z and $(1-z)$ are finite, we have $q_1 = (1-q)z$ and $p_1 = (1-p)(1-z)$. Thus, $1-q \sim 1-p$. □

Definition: Let M be a von Neumann algebra.

- We say M is type I if every nonzero central projection in M majorizes a nonzero abelian projection in M .
- We say M is type II_1 if it is finite, has no nonzero abelian projections, and every nonzero central projection in M majorizes a nonzero finite projection.
- We say M is type II_∞ if every nonzero central projection majorizes a nonzero finite projection, and has no nonzero finite central projections.
- We say M is type III if it is purely infinite.

Theorem: Every von Neumann algebra M uniquely decomposes into a direct sum of those of type I, II_1 , II_∞ , and III. Moreover, every projection e in M can be uniquely written as the sum of centrally orthogonal pro-

jections e_1 and e_2 in M such that e_1 is finite and e_2 is properly infinite.

Proof. Let $\{e_i\}_{i \in I}$ be a maximal family of centrally orthogonal abelian projections in M , and let $e = \sum_{i \in I} e_i$. Then, e is abelian; define $z_I = z(e)$.

If z is a nonzero central projection majorized by z_I , then ze is a nonzero abelian projection, so Mz_I is of type I.

By construction, there is then no nonzero abelian projection in $M(1 - z_I)$, so it has no nontrivial direct summand of type I.

Let $\{f_j\}_{j \in J}$ be a maximal family of centrally orthogonal finite projections in $M(1 - z_I)$, and let $f = \sum_{j \in J} f_j$. Then, f is finite. Set $z_{II} = z(f)$. By construction, we have that Mz_{II} has no nonzero abelian projections, and every nonzero projection z in Mz_{II} majorizes a nonzero finite projection zf . Thus, Mz_{II} is of type II.

By the maximality of $\{f_j\}_{j \in J}$, it follows that $1 - z_{II} = z_{III}$ does not majorize any finite projection, so Mz_{III} is of type III. We have that $z_I + z_{II} + z_{III} = 1$.

Now, let $\{z_k\}_{k \in K}$ be a maximal orthogonal family of finite central projections in Mz_{II} , and set $z_{II_1} = \sum_{k \in K} z_k$, and $z_{II_\infty} = z_{II} - z_{II_1}$. It follows that Mz_{II_1} is of type II_1 and Mz_{II_∞} is of type II_∞ . Therefore, we get the direct sum decomposition

$$M = Mz_I \oplus Mz_{II_1} \oplus Mz_{II_\infty} \oplus Mz_{III}.$$

As for uniqueness, we suppose that there is another orthogonal decomposition $1 = w_I + w_{II_1} + w_{II_\infty} + w_{III}$. Then, we must have that $w_I(1 - z_I) = 0$ since $1 - z_I$ does not majorize any nonzero abelian projection, while $w_I(1 - z_I)$ is a central projection in Mw_I . Therefore, $w_I \leq z_I$. Similarly, $z_I(1 - w_I) = 0$ for the same reason, meaning $z_I \leq w_I$, so they are equal. By similar arguments, all the other summands are equal to each other.

Finally, we let e be a nonzero projection in M . By considering $M_e := eMe$, we may assume that $e = 1$. We let e_1 be the sum of a maximal orthogonal family of finite central projections. Then, e_1 is finite and central, with $1 - e_1$ properly infinite. The uniqueness of this decomposition has the same flavor as the arguments for $\{z_I, \dots, z_{III}\}$. $\square$

Definition: We say a factor is atomic if it contains a minimal projection, and otherwise we say it is diffuse.

Structure of Type I and II von Neumann Algebras

Now, we will discuss some structural results related to type I and II von Neumann algebras. Before we can do this, we must discuss tensor products.

Tensor Products of Hilbert Spaces and Operators

Definition: Let H and K be Hilbert spaces. There is an inner product on the algebraic tensor product $H \odot K$ given by

$$\langle \xi_1 \otimes \xi_2, \eta_1 \otimes \eta_2 \rangle = \langle \xi_1, \eta_1 \rangle \langle \xi_2, \eta_2 \rangle,$$

whenever $\xi_1, \eta_1 \in H$ and $\xi_2, \eta_2 \in K$. The Hilbert space tensor product of H and K , denoted $H \otimes K$, is the completion of $H \odot K$ with respect to the norm induced by this inner product.

Proposition: Let H and K be Hilbert spaces with orthonormal bases $\{e_i\}_{i \in I}$ and $\{f_j\}_{j \in J}$. Then,

- (i) $\{e_i \otimes f_j\}_{i \in I, j \in J}$ is an orthonormal basis for $H \otimes K$;
- (ii) if $|J| = \alpha$ for some cardinal α , then $H \otimes K \cong H^{(\alpha)} \cong \bigoplus_{j \in J} H$;
- (iii) if $H = L_2(X, \mu)$ for some σ -finite regular Borel measure space (X, μ) , and K is separable, then $H \otimes K \cong L_2(X, \mu, K)$, where the latter denotes the space of square-integrable Borel functions with respect to the

norm on K .

Proof.

(i) We observe that the set $\{e_i \otimes f_j\}_{i \in I, j \in J}$ is an orthonormal set. The spans of these elementary tensors are all the vectors of the form $x \otimes y$, provided that x is a finite linear combination of the e_i and y is a finite linear combination of the f_j . Therefore, the completion is equal to the completion of $H \otimes K$, so the set is an orthonormal basis.

(ii) We find that

$$H \otimes K \cong \bigoplus_{j \in J} H \otimes \mathbb{C}f_j$$

is an ℓ_2 direct sum of $|J|$ copies of H .

(iii) Define maps $Y_j: L_2(X, \mu, K) \rightarrow L_2(X, \mu)$ by taking

$$(Y_j f)(x) = \langle f(x), f_j \rangle,$$

a representative of an equivalence class modulo μ . Then, we have that

$$f \mapsto \sum_{j \in J} Y_j f$$

defines a map from $L_2(X, \mu, K) \rightarrow \bigoplus_{j \in J} H \otimes \mathbb{C}f_j$. By Tonelli's theorem and Parseval's identity, we get

$$\begin{aligned} \|Y f\|^2 &= \sum_{j \in J} \|Y_j f\|^2 \\ &= \sum_{j \in J} \int |\langle f(x), f_j \rangle|^2 d\mu \\ &= \int \sum_{j \in J} |\langle f(x), f_j \rangle|^2 d\mu \\ &= \int \|f(x)\|^2 d\mu \\ &= \|f\|^2. \end{aligned}$$

Therefore, Y is an isometry. The range is dense since, if $h \in L_2(\mu)$, then $f := hf_j$ is mapped to the vector with h in position j and 0 elsewhere. Thus, Y is a unitary map from $L_2(X, \mu, K)$ onto $H \otimes K$. $\square$

Proposition: Let $A \in B(H)$ and $B \in B(K)$. Then, there is a unique $A \otimes B \in B(H \otimes K)$ such that $(A \otimes B)(\xi \otimes \eta) = A\xi \otimes B\eta$. Furthermore, $\|A \otimes B\| = \|A\| \|B\|$.

Proof. Let $A \otimes I_K$ be the amplification to $H \otimes K$. Fixing an orthonormal basis $\{\eta_j\}_{j \in J}$ for K , then any vector $\omega \in H \otimes K$ (from part (ii) above) can be expressed as

$$\omega = (\zeta_j \otimes \eta_j)_j$$

for a collection of $\zeta_j \in H$, and we may define

$$\begin{aligned} \|\omega\|^2 &= \sum_{j \in J} \|\zeta_j\|^2 \\ &< \infty \end{aligned}$$

Therefore, we may rewrite

$$(A \otimes I)\omega = (A\zeta_j \otimes \eta_j)_j,$$

giving

$$\begin{aligned} \|(A \otimes I)\omega\|^2 &= \left\| (A\zeta_j \otimes \eta_j)_j \right\|^2 \\ &= \sum_{j \in I} \|A\zeta_j\|^2 \\ &\leq \|A\|^2 + \sum_{j \in I} \|\zeta_j\|^2 \\ &= (\|A\| \|\omega\|)^2. \end{aligned}$$

It follows that $\|A \otimes I\| = \|A\|$. Similarly, we may define $I \otimes B$, and set $A \otimes B = (A \otimes I)(I \otimes B)$. Thus, we have

$$\begin{aligned} \|A \otimes B\|(\xi \otimes \eta) &= (A \otimes I)(I \otimes B)(\xi \otimes \eta) \\ &= (A \otimes I)(\xi \otimes B\eta) \\ &= A\xi \otimes B\eta. \end{aligned}$$

We have that $\|A \otimes B\| \leq \|A \otimes I\| \|I \otimes B\| = \|A\| \|B\|$. Since $\|(A \otimes B)(\xi \otimes \eta)\| = \|A\xi\| \|B\eta\|$, we may choose unit vectors ξ and η appropriately to approximate $\|A\| \|B\|$. Finally, since $H \odot K$ is dense in $H \otimes K$, we have that $A \otimes B$ is well-defined. $\square$

Definition: Let M and N be von Neumann algebras. Then, the von Neumann algebra tensor product $M \bar{\otimes} N$ is the WOT-closure of $M \odot N$ in $B(H \otimes K)$. If $N = \mathbb{C}1$, then $M \bar{\otimes} \mathbb{C}1 = M^{(\alpha)}$ is an amplification of M .

Definition: A system $\{w_{ij} \mid i, j \in I\}$ of elements in a von Neumann algebra is called a system of *matrix units* if

- (i) $w_{ij}^* = w_{ji}$;
- (ii) $w_{ij}w_{k\ell} = \delta_{jk}w_{i\ell}$;
- (iii) $\sum_{i \in I} w_{ii} = 1$.

Proposition: Let $M \subseteq B(H)$ be a von Neumann algebra. Let $\{w_{ij} \mid i, j \in I\}$ be a system of matrix units. Fix $i_0 \in I$, and let $e = w_{i_0 i_0}$. Then,

$$M \cong (eMe) \otimes B(\ell_2(I)).$$

Proof. Let $\{\varepsilon_i\}_{i \in I}$ be an orthonormal basis in $\ell_2(I)$, and let $v_i = w_{i, i_0}$ for each $i \in I$. Then, $w_{ij} = v_i v_j^*$ and $v_i^* v_i = e$. Define a map

$$U \left(\sum_{i \in I} \xi_i \otimes \varepsilon_i \right) = \sum_{i \in I} v_i \xi_i$$

for each $\sum_{i \in I} \xi_i \otimes \varepsilon_i \in K \otimes \ell_2(I)$. Then, we have

$$\begin{aligned} \left\langle U \left(\sum_{i \in I} \xi_i \otimes \varepsilon_i \right), U \left(\sum_{j \in I} \eta_j \otimes \varepsilon_j \right) \right\rangle &= \left\langle \sum_{i \in I} v_i \xi_i, \sum_{j \in I} v_j \eta_j \right\rangle \\ &= \sum_{i, j \in I} \langle v_j^* v_i \xi_i, \eta_j \rangle \\ &= \sum_{i \in I} \langle \xi_i, \eta_i \rangle \end{aligned}$$

$$= \left\langle \sum_{i \in I} \xi_i \otimes \varepsilon_i, \sum_{j \in I} \eta_j \otimes \varepsilon_j \right\rangle,$$

so that U is a well-defined isometry of $K \otimes \ell_2(I)$ onto H . Since the range of U contains all $v_i K = w_{ii} H$, it follows that U is surjective as $\sum_{i \in I} w_{ii} = 1$.

Now, letting $x = (x_{ij}) \in eMe \otimes B(\ell_2(I))$, we have

$$\begin{aligned} UxU^*\xi &= Ux \left(\sum_{j \in I} v_j^* \xi \otimes \varepsilon_j \right) \\ &= U \left(\sum_{i,j \in I} x_{ij} v_j \xi \otimes \varepsilon_i \right) \\ &= \sum_{i,j \in I} v_i x_{ij} v_j^* \xi, \end{aligned}$$

meaning that $UxU^* = \sum_{i,j \in I} v_i x_{ij} v_j^*$ is an element of M . Meanwhile, if $x \in M$, then

$$\begin{aligned} U^*xu \left(\sum_{j \in I} \xi_j \otimes \varepsilon_j \right) &= U^*x \sum_{j \in I} v_j \xi_j \\ &= U^* \sum_{j \in I} xv_j \xi_j \\ &= \sum_{i,j \in I} (v_i^* xv_j \xi_j \otimes \varepsilon_i). \end{aligned}$$

Therefore, $(U^*xU)_{i,j} = v_i^* xv_j$, so that $U^*xU \in eMe \bar{\otimes} B(\ell_2(I))$. $\square$

It turns out that the tensor product is part of a general theory of representations of von Neumann algebras.

Lemma: Let $M \subseteq B(H)$ be a von Neumann algebra, where H is a separable infinite-dimensional Hilbert space. For any $\omega \in M_*$, where M_* denotes the normal (σ -WOT continuous) linear functionals on M , there exist vectors $\xi, \eta \in H \otimes H$ such that

$$\omega(x) = \langle (x \otimes 1)\xi, \eta \rangle.$$

Proof. Let $\{\varepsilon_n\}_{n \in \mathbb{N}}$ be an orthogonal basis for H . By the structure of σ -WOT continuous linear functionals, we have

$$\omega = \sum_{n=1}^{\infty} \theta_{\xi_n, \eta_n},$$

where

$$\theta_{x,y}(z) = \langle z, y \rangle x$$

is a rank-one projection, and the sequences $(\xi_n)_n$ and $(\eta_n)_n$ are square-summable in norm. If we set

$$\begin{aligned} \xi &= \sum_{n=1}^{\infty} \xi_n \otimes \varepsilon_n \\ \eta &= \sum_{n=1}^{\infty} \eta_n \otimes \varepsilon_n, \end{aligned}$$

then $\xi, \eta \in H \otimes H$, and satisfies

$$\begin{aligned}\omega(x) &= \sum_{n=1}^{\infty} \langle x \xi_n, \eta_n \rangle \\ &= \sum_{n,m=1}^{\infty} \langle (x \otimes 1)(\xi_n \otimes \varepsilon_n), (\eta_m \otimes \varepsilon_m) \rangle \\ &= \langle (x \otimes 1)\xi, \eta \rangle.\end{aligned}$$

□

Analogously, any normal homomorphism of a von Neumann algebra can be decomposed into components including a tensor product.

Theorem: Let $M_1 \subseteq B(H_1)$ and $M_2 \subseteq B(H_2)$ be von Neumann algebras. If π is a normal homomorphism from M_1 to M_2 , then there is a Hilbert space H , a projection $e' \in M_1' \otimes B(H)$, and a unitary $U: e'(H_1 \otimes H) \rightarrow H_2$ such that

$$\pi(x) = U e' (x \otimes 1_H) e' U^*$$

for all $x \in M_1$. That is, every normal homomorphism between M_1 and M_2 consists of an amplification, a compression, and a spatial isomorphism.

Proof. Let $\{\xi_{2,i}\}_{i \in I}$ be a maximal family of nonzero vectors in H_2 such that the range subspaces $[M_2 \xi_{2,i}] = H_{2,i}$ are mutually orthogonal. From maximality, we have that

$$H_2 = \bigoplus_{i \in I} H_{2,i}.$$

Now, let

$$H = \bigoplus_{i \in I} H_i,$$

where $H_i \cong H_0$ for some fixed separable infinite-dimensional Hilbert space H_0 . Let π_1 be the amplification of M_1 to $M_1 \otimes 1_H$. If $\{\varepsilon_n\}_{n \in \mathbb{N}}$ is an orthogonal basis of H_0 , then by the above lemma, we may find a vector $\xi_{1,i} \in H_1 \otimes H_i$ such that

$$\langle \pi(x) \xi_{2,i}, \xi_{2,i} \rangle = \langle \pi_1(x) \xi_{1,i}, \xi_{1,i} \rangle$$

for each $x \in M_1$.

Observe that for each i , the ranges $[(M_1 \otimes \mathbb{C}1_H) \xi_{1,i}]$ are mutually orthogonal in $H_1 \otimes H$.

Define e' to be the range projection onto $\bigoplus_{i \in I} [(M_1 \otimes \mathbb{C}1_H) \xi_{1,i}]$. Then, we have that $e' \in (M_1 \otimes \mathbb{C}1_H)' = M_1' \otimes B(H)$. Then, for each $i \in I$, we may define

$$U_i \pi_1(x) \xi_{1,i} = \pi(x) \xi_{2,i}.$$

By the definition of $\xi_{1,i}$, it follows that U_i may be extended to an isometry of $[\pi_1(M_1) \xi_{1,i}]$ onto $H_{2,i}$, which we also denote by U_i . Define

$$U = \bigoplus_{i \in I} U_i.$$

If we let $\pi_2(x) = e' x e'$ for any $x \in M_1 \otimes \mathbb{C}1_H$, and $\pi_3(x) = U x U^*$ for any $x \in e'(M_1 \otimes \mathbb{C}1_H) e'$, then we have the desired decomposition of π . □

Type I von Neumann Algebras

We start by considering the structure and classification of type I von Neumann algebras. For starters, we know that every projection dominates a rank one projection (which is minimal), so since any minimal projection is abelian, it follows that $B(H)$ is a factor of type I. Furthermore, we can show that any atomic von Neumann algebra is type I.

Proposition:

- (i) If A is an abelian von Neumann algebra, then $A \bar{\otimes} B(H)$ is type I for any Hilbert space H .
- (ii) If A is an abelian von Neumann algebra, then the matrix algebra $M_n(A)$ is a finite von Neumann algebra of type I, and the A -valued map

$$\Phi(x) = \frac{1}{n} \sum_{i=1}^n x_{ii}$$

is

- center-valued: $\Phi\left(\left(\delta_{ij}a\right)_{i,j}\right) = a$ whenever $a \in A$;
- A -modular: $\Phi(axb) = a\Phi(x)b$ whenever $a, b \in A$;
- positive definiteness: $\Phi(x^*x) \geq 0$, with equality only when $x = 0$;
- traciality: $\Phi(x^*x) = \Phi(xx^*)$.

Proof.

- (i) Let p be a minimal projection in $B(H)$, and let $e = 1 \otimes p$. Then,

$$\begin{aligned} e(A \bar{\otimes} B(H))e &= A \bar{\otimes} pB(H)p \\ &= A \otimes \mathbb{C}p, \end{aligned}$$

so e is an abelian projection. Since the center of $A \bar{\otimes} B(H)$ is $A \otimes \mathbb{C}$, it follows that the central support $z(e)$ of e is the identity. Thus, $A \bar{\otimes} B(H)$ is type I.

- (ii) The first two properties of Φ follow from the definition, while the latter two properties follow from taking

$$\begin{aligned} \Phi(x^*x) &= \frac{1}{n} \sum_{i,j=1}^n x_{ij}^* x_{ij} \\ &= \frac{1}{n} \sum_{i,j=1}^n x_{ij} x_{ij}^* \\ &= \Phi(xx^*). \end{aligned}$$

□

Definition: An A -valued map satisfying the properties in (ii), except for possibly the faithfulness condition, is known as an A -valued trace.

We will discuss traces later. First, we will discuss further characterizations and structures of type I von Neumann algebras.

Proposition: A von Neumann algebra is type I if and only if it has an abelian projection whose central support is 1. We call such projections faithful.

Proof. Let e be a faithful abelian projection in M , and let g be any nonzero central projection. Then, $eg \neq 0$. Yet, $(eg)M(eg) = (eMe)g$, which is abelian; since $eg \leq g$, it follows that M is type I.

Now, if M is type I, then by Zorn's lemma, there is a maximal family of centrally orthogonal abelian projections $\{e_i\}_{i \in I}$. Set $z = \sum_i z(e_i)$. Then, $(1 - z)M(1 - z)$ is type I, so if $1 - z \neq 0$ we get a contradiction by the maximality of $\{e_i\}_{i \in I}$. Thus, $\sum_{i \in I} z(e_i) = 1$.

If we let $e = \sum_{i \in I} e_i$, then e is abelian and $z(e) = 1$. $\square$

Definition: Let n be a cardinal number. We say a type I von Neumann algebra is type I_n if 1 is a sum of n equivalent nonzero abelian projections. We say it is type I_∞ if it is type I_n for an infinite cardinal n .

Furthermore, if z is any projection that is the sum of n equivalent abelian projections with central support equal to z , then we say z is n -homogeneous.

Lemma: Let M be a von Neumann algebra, and let $p \in P(M)$ be a projection with central support $z(p)$. Then,

$$M'z(p) \rightarrow M'p$$

given by $yz(p) \mapsto yp$ is an isomorphism of $*$ -algebras. In particular, $Z(pMp) = Z(Mz(p))p$.

Proof. Recalling that $z(p) = [MpH]$, we have that if $yp = 0$, then for all $x \in M$ and $\xi \in H$,

$$\begin{aligned} yxp\xi &= xyp\xi \\ &= 0. \end{aligned}$$

Therefore, $yz(p) = 0$, meaning that $yz(p) \mapsto yp$ is injective. Since this map is surjective, it follows that we get this as an isomorphism.

In particular, we have that $Z(pMp) = Z(M'p)$, and since $yp \in Z(M'p)$ if and only if $yz(p) \in Z(M'z(p))$, we find our desired equality. $\square$

An important fact about abelian projections is that their central supports fully encode their order relation (in the following sense).

Lemma: Let $M \subseteq B(H)$ be a von Neumann algebra. If $p, q \in P(M)$ are abelian projections with $z(p) = z(q)$, then $p \sim q$.

Note that we already have that if $p \sim q$, then $z(p) = z(q)$, but this provides a type of converse.

Proof. Suppose $p \leq q$, and let $v \in M$ implement this subequivalence, where $vv^* = q_0 \leq q$. Since q is abelian, we have

$$\begin{aligned} qMq &= Z(qMq) \\ &= Z(Mz(q))q, \end{aligned}$$

so since $q_0 \in qMq$, we have $q_0 = zq$ for some central projection $z \in Z(Mz(q))$. Therefore, we get

$$\begin{aligned} z(p) &= z(q_0) \\ &= zz(q) \\ &= z(q) \\ &= z(p). \end{aligned}$$

In particular, this gives $zz(q) = z(q)$, so $z(q) \leq z$, and thus $z = z(q)$, $q_0 = q$, and $p \sim q$.

In the case that p is not subequivalent to q , then by the comparison theorem, there is $z \in Z(M)$ such that $pz \leq qz$ and $q(1 - z) \leq p(1 - z)$. Now, we have $z(pz) = z(p)z = z(qz)$, so $pz \sim qz$, and $q(1 - z) \sim p(1 - z)$, so $p \sim q$. $\square$

Lemma: Let $M \subseteq B(H)$ be a von Neumann algebra. If n and m are cardinals such that M is type I_n and I_m , then $n = m$.

Proof. Let $\{p_i\}_{i \in I}$ and $\{q_j\}_{j \in J}$ be pairwise orthogonal equivalent abelian projections with $|I| = n$ and $|J| =$

m such that

$$\begin{aligned} 1 &= \sum_{i \in I} p_i \\ &= \sum_{j \in J} q_j. \end{aligned}$$

Note that $z(p_i)$ is constant across i , so that

$$\begin{aligned} 1 - z(p_i) &= (1 - z(p_i)) \sum_{k \in I} p_k \\ &= 0, \end{aligned}$$

so that $z(p_i) = 1$ for all $i \in I$. Similarly, $z(q_j) = 1$ for all $j \in J$, meaning $p_i \sim q_j$ for all $i \in I$ and $j \in J$.

Now, we start by assuming that $n < \infty$. For each $i \in I$ and each $j \in J$, let $v_i^* v_i = u_j^* u_j = p_{i_0}$, $v_i v_i^* = p_i$, and $u_j u_j^* = q_j$. Now, consider the map

$$\Phi(x) = \frac{1}{n} \sum_{i \in I} v_i^* x v_i.$$

Then,

$$\begin{aligned} \Phi(xy) &= \frac{1}{n} \sum_{i \in I} v_i^* x y v_i \\ &= \frac{1}{n} \sum_{i, k \in I} v_i^* x v_k v_k^* y v_i \\ &= \frac{1}{n} \sum_{i, k \in I} (p_{i_0} v_i^* x v_k p_{i_0}) (p_{i_0} v_k^* y v_i p_{i_0}) \\ &= \frac{1}{n} \sum_{i, k \in I} (p_{i_0} v_i^* y v_k p_{i_0}) (p_{i_0} v_k^* x v_i p_{i_0}) \\ &= \Phi(yx). \end{aligned}$$

Consequently, $\Phi(p_i) = \Phi(q_j) = \Phi(p_{i_0})$ for all $i \in I$ and all $j \in J$, with common image p_{i_0} . Therefore, we get

$$\begin{aligned} np_{i_0} &= \sum_{i \in I} \Phi(p_i) \\ &= \Phi(1) \\ &= \sum_{j \in J} \Phi(q_j) \\ &= mp_{i_0}, \end{aligned}$$

so $n = m$.

Now, suppose n is infinite. By a symmetric argument, we have m is infinite. Let φ be a normal state with support $z \in Z(M)$; that is, $w \in \ker(\varphi)$ if and only if $zwz = 0$. For each $i \in I$, define $\sigma_i: Z(M) \rightarrow p_i M p_i$ by $\sigma_i(z) = zp_i$. This is an isomorphism from the earlier lemma, so we may define φ_i by

$$\varphi_i(x) = \varphi(\sigma_i^{-1}(p_i x p_i)).$$

We have that φ_i is a state with support zp_i . Since σ_i is σ -WOT continuous, we have

$$1 = \varphi_i(1)$$

$$\begin{aligned}
&= \varphi_i \left(\sum_{j \in J} q_j \right) \\
&= \sum_{j \in J} \varphi_i(q_j).
\end{aligned}$$

Since $\varphi_i(q_j) \geq 0$ for each $j \in J$, the set of all j for which $\varphi_i(q_j) > 0$ is countable. Yet, if $j \notin J_i$, we have $\varphi_i(q_j) = 0$, meaning $(zp_i)q_j(zp_i) = 0$, and by centrality, we have $zp_iq_jp_i z = 0$, so that $q_jp_i z = 0$.

Yet, for any $j \in J$, we have

$$\sum_{i \in I} q_j p_i z = q_j z,$$

which is nonzero as $z(q_j) = 1$. In particular, this means $q_j p_i z \neq 0$ for some $i \in I$, meaning $j \in J_i$ for some $i \in I$. That is, $J = \bigcup_{i \in I} J_i$, so that $m \leq n \aleph_0 = n$, and by symmetry, $m = n$. $\square$

Theorem: Let M be a von Neumann algebra of type I. Then, there exists a unique family of orthogonal central projections $\{z_\alpha\}_\alpha$ indexed by cardinal numbers such that $\sum_\alpha z_\alpha = 1$, and $z_\alpha M z_\alpha$ is isomorphic to the tensor product of an abelian von Neumann algebra A_α and $B(H_\alpha)$, where $\dim(H_\alpha) = \alpha$. In particular, we have

$$M \cong \bigoplus_\alpha A_\alpha \bar{\otimes} B(H_\alpha).$$

Proof. We note that if $\{z_i\}_{i \in I}$ is a family of α -homogeneous projections, then their sum is also α -homogeneous. Therefore, upon taking suprema, we may find a greatest z_α that is α -homogeneous.

If $z_\alpha \neq 0$, then there exists an orthogonal family $\{e_i\}_{i \in I_\alpha}$ of abelian projections with $z(e_i) = z_\alpha$, and I_α has cardinality α . Since these projections are mutually equivalent, we have

$$z_\alpha M z_\alpha \cong e_1 M e_1 \bar{\otimes} B(\ell_2(I_\alpha)).$$

Since $e_1 M e_1$ is abelian, we only have to show that z_α and z_β are orthogonal for distinct α and β .

If α is infinite, then both I_α and $I_\alpha \setminus \{i_0\}$ have the same cardinality, meaning that $z_\alpha = \sum_{i \in I_\alpha} e_i \sim \sum_{i \in I_\alpha \setminus \{i_0\}} e_i < z_\alpha$, meaning z_α is not a finite projection. In particular, α must be finite if M is finite. $\square$

Proposition: A von Neumann algebra M is type I if and only if M admits a faithful normal representation $\pi: M \rightarrow B(H)$ such that $\pi(M)'$ is abelian.

Proof. Let $M \subseteq B(H)$ be a von Neumann algebra with abelian commutant M' . Then, M' coincides with $Z(M)$. For each $\xi \in H$, let e_ξ denote the projection of H onto $[Z(M)\xi]$. Since e_ξ commutes with $Z(M)$, it follows that e_ξ is contained in M . Since $e_\xi Z(M) e_\xi \subseteq B([Z(M)\xi]H)$ admits a cyclic vector, namely ξ , it follows that $e_\xi Z(M) e_\xi$ is abelian, so $(e_\xi Z(M) e_\xi)' = e_\xi M e_\xi$ is abelian. Thus, e_ξ is abelian for any $\xi \in H$. Furthermore, for any projection $p \in M$, we have that $e_\xi \leq p$ whenever $\xi \in pH$, so every nonzero projection majorizes an abelian projection, so M is type I.

Now, if M is type I, then we may represent M as

$$M \cong \bigoplus_\alpha A_\alpha \bar{\otimes} B(H_\alpha)$$

and take A_α as a masa of $B(H_\alpha)$, meaning that on the Hilbert space

$$H := \bigoplus_\alpha H_\alpha \otimes H_\alpha,$$

we have

$$\begin{aligned} M' &= \left(\bigoplus_{\alpha} A'_{\alpha} \bar{\otimes} B(H_{\alpha}) \right)' \\ &= \bigoplus_{\alpha} (A_{\alpha} \otimes \mathbb{C}) \\ &= \bigoplus_{\alpha} A_{\alpha}, \end{aligned}$$

so M' is abelian. □

Corollary: If M is a von Neumann algebra of type I, then so is M' .

Traces on Finite von Neumann Algebras

Lemma: Let M be a finite von Neumann algebra, and $p \in P(M)$ a nonzero projection. If we have $\{p_i\}_{i \in I}$ a family of pairwise orthogonal projections with $p_i \sim p$ for each i , then I is finite.

Proof. If this were not the case, then we would have a proper subset $J \subsetneq I$ with $\sum_{i \in I} p_i \sim \sum_{j \in J} p_j < \sum_{i \in I} p_i$, implying $\sum_{i \in I} p_i$ is not finite, but the sum of pairwise orthogonal finite projections is finite. □

Lemma: Let $M \subseteq B(H)$ be a type II₁ von Neumann algebra. Then, there exists a projection $p_{1/2} \in P(M)$ such that $p_{1/2} \sim 1 - p_{1/2}$.

Furthermore, there is a family of projections $\{p_r\}_r$ indexed by dyadic rationals in $[0, 1]$ such that $p_r \leq p_s$ if $r \leq s$, $p_s - p_r \sim p_{s'} - p_{r'}$ whenever $0 \leq r \leq s \leq 1$, $0 \leq r' \leq s' \leq 1$, and $s - r = s' - r'$, and each of the p_r has central support equal to 1.

Proof. Let $\{p_i, q_i\}_{i \in I}$ be a maximal family of pairwise orthogonal projections with $p_i \sim q_i$ for all $i \in I$. Define

$$\begin{aligned} p_{1/2} &= \sum_{i \in I} p_i \\ q &= \sum_{i \in I} q_i. \end{aligned}$$

We have that $p_{1/2} \sim q$, and we claim that $q \sim 1 - p_{1/2}$. If not, then $1 - (p_{1/2} + q) \neq 0$, so since M is type II, it follows that $1 - (p_{1/2} + q)$ is not abelian, so there exists some projection in the corner of M with respect to $1 - (p_{1/2} + q)$ that is strictly less than its central support in the corner. We denote this central support by z . Then, if we set $q_0 = z - p_0$, we have that p_0 and q_0 are not centrally orthogonal, so they have equivalent subprojections. Yet, this contradicts maximality of $\{p_i, q_i\}_{i \in I}$, so it must be the case that $q = 1 - p_{1/2}$.

Now, we construct the family of projections indexed by the dyadic rationals inductively. Let $p_{1/2}$ be defined as above, $p_0 = 0$, and $p_1 = 1$. If $v \in M$ is such that $v^*v = p_{1/2}$ and $vv^* = 1 - p_{1/2}$. Then, $p_{1/2}Mp_{1/2}$ is type II, and specifically type II₁, since $p_{1/2}$ is a finite projection. If we let $q \sim p_{1/2}$ with $q < p_{1/2}$, then $q + (1 - p_{1/2}) \sim 1$ with $q + (1 - p_{1/2}) < 1$, which contradicts the finitude of 1. Therefore, we may obtain $p_{1/4} \leq p_{1/2}$ with $p_{1/4} \sim p_{1/2} - p_{1/4}$, and set $p_{3/4} = p_{1/2} + vp_{1/2}v^*$, with $p_0 \leq p_{1/4} \leq p_{1/2} \leq p_{3/4} \leq p_1$, with $p_{1/4} \sim p_{(k+1)/4} - p_{k/4}$ for each $k = 0, 1, 2, 3$. Thus, inductively, we obtain a family indexed by dyadic rationals satisfying $p_s - p_r \sim p_{s'} - p_{r'}$ whenever $s - r = s' - r'$.

Now, we verify central supports. Fix a dyadic rational r , and set $z = 1 - z(p_r)$. Let $n \in \mathbb{N}$ be large enough that $s = \frac{1}{2^n} \leq r$. Then, we have that $zp_s \leq zp_r = 0$, and $zp_s \sim z(p_{ks} - p_{(k-1)s})$ for every $1 \leq k \leq 2^n$, meaning that $z(p_{ks} - p_{(k-1)s}) = 0$. Thus,

$$\begin{aligned} z &= z \sum_{k=1}^n (p_{ks} - p_{(k-1)s}) \\ &= 0, \end{aligned}$$

so $z(p_r) = 1$. □

Lemma: Let M be a type II_1 von Neumann algebra, and let $\{p_r\}_r$ be the above collection of projections. If $p \in P(M)$, with $p \neq 0$, then there exists $z \in P(Z(M))$ such that $pz \neq 0$ and $p_r z \leq pz$ for some positive dyadic rational r .

Proof. Passing to $Mz(p)$ allows us to assume that $z(p) = 1$. If this does not hold, then we would have $p \leq p_r$ for every $r > 0$, meaning that p would be equivalent to a subprojection of $p_{2^{-k}} - p_{2^{-(k+1)}}$, which contradicts the fact that any family of pairwise orthogonal projections equivalent to a fixed projection is finite. $\square$

We say a projection $p \in P(M)$ is *monic* if there exists a finite collection of pairwise orthogonal projections $\{p_1, \dots, p_n\}$ such that $p_i \sim p$ and $\sum_{i=1}^n p_i \in Z(M)$. The projections $p_{1/2^k}$ defined above in the II_1 case are monic.

Proposition: If M is a finite von Neumann algebra, then every projection is the sum of pairwise orthogonal monic projections.

Proof. By passing to a maximal sub-family, it suffices to show that every nonzero projection has a nonzero monic subprojection. Furthermore, by passing to the type decomposition, it suffices to consider the cases where M is type II_1 or M is type I_n .

In the case of type II_1 , we get this from the above discussion, and in the case of type I_n , this follows from considering a nonzero abelian subprojection. $\square$

Definition: Let M be a von Neumann algebra. We say a linear map $\varphi: M \rightarrow Z(M)$ is a *center-valued state* if $\varphi(x^*x) \geq 0$, $\varphi|_{Z(M)} = \text{id}$, and $\varphi(zx) = z\varphi(x)$ whenever $z \in Z(M)$ and $x \in M$.

Lemma: If M is a von Neumann algebra and $\varphi: M \rightarrow Z(M)$ is a center-valued state, then φ is bounded with $\|\varphi\| = 1$.

Lemma: Let $M \subseteq B(H)$ be a von Neumann algebra. Then, M has a normal center-valued state.

Proof. Since $Z(M)$ is an abelian von Neumann algebra, $Z(M)'$ is type I , so it has an abelian projection q with full central support. Then, $qMq \subseteq qZ(M)'q = Z(M)q$, so $\theta(z) = zq$ defines a normal isomorphism from $Z(M)$ onto $Z(M)q$. Setting $\varphi(x) = \theta^{-1}(qxq)$, then φ is a normal center-valued state. $\square$

Lemma: Let M be a von Neumann algebra, and let $\tau: M \rightarrow Z(M)$ be a center-valued state. The following are equivalent:

- (i) $\tau(xy) = \tau(yx)$ for all $x, y \in M$;
- (ii) $\tau(xx^*) = \tau(x^*x)$ for all $x \in M$;
- (iii) $\tau(p) = \tau(q)$ for all equivalent projections $p, q \in P(M)$.

Proof. We only need to show (iii) implies (i). First, for all $p \in P(M)$ and all unitaries u , we have $\tau(upu^*) = \tau(p)$. Since τ is bounded, then by applying the functional calculus, we have that $\tau(uxu^*) = \tau(x)$ for all self-adjoint $x \in M$. Separating real and imaginary parts, and replacing x with xu , it follows that $\tau(ux) = \tau(xu)$ for all $x \in M$ and all unitaries u . Since every operator is a linear combination of four unitaries, this shows (i). $\square$

As discussed above, such a center-valued state is known as a center-valued trace.

Lemma: Let $\varphi: M \rightarrow Z(M)$ be a normal center-valued state. Then, for every $\varepsilon > 0$, there exists $p \in P(M)$ with $\varphi(p) \neq 0$ and for all $x \in pMp$,

$$\varphi(xx^*) \leq (1 + \varepsilon)\varphi(x^*x).$$

Proof. Let $q_0 = 1 - \sum_{i \in I} q_i$, where $\{q_i\}_{i \in I}$ is a maximal family of pairwise orthogonal projections with $\varphi(q_0) = 0$. By normality, we must have that $\varphi(q_0) = 1$, and φ is faithful when restricted to $q_0 M q_0$.

Let $\{e_i, f_i\}_{i \in I}$ be a maximal family of projections with $\{e_i\}_{i \in I}$ and $\{f_i\}_{i \in I}$ pairwise orthogonal, $e_i \sim f_i$ for each i , and $\varphi(e_i) > \varphi(f_i)$ for each i . Setting $e = q_0 - \sum_{i \in I} e_i$ and $f = q_0 - \sum_{i \in I} f_i$, then unless φ is already a trace, we have $\varphi(f) > \varphi(e) \geq 0$. Hence, $f \neq 0$, so $e \sim f$, meaning $e \neq 0$.

Let μ be the smallest number such that $\varphi(\bar{e}) \leq \mu\varphi(\bar{f})$ whenever $\bar{e} \leq e$, $\bar{f} \leq f$, and $\bar{e} \sim \bar{f}$. Then, $\mu \neq 0$ since $\varphi(e) \neq 0$, and there exists $\bar{e} \leq e$, $\bar{f} \leq f$ such that $\bar{e} \sim \bar{f}$, $(1 + \varepsilon)\varphi(\bar{e}) \not\leq \mu\varphi(\bar{f})$. By cutting by a suitable cen-

tral projection, we may assume that $(1 + \varepsilon)\varphi(\bar{e}) > \mu\varphi(\bar{f})$.

Let $\{\hat{e}_i, \hat{f}_i\}_{i \in I}$ be a maximal family with $\{\hat{e}_i\}_{i \in I}$ and $\{\hat{f}_i\}_{i \in I}$ are pairwise orthogonal. Suppose $\hat{e}_i \leq \bar{e}$, $\hat{f}_i \leq \bar{f}$, and $\hat{e}_i \sim \hat{f}_i$ with $(1 + \varepsilon)\varphi(\hat{e}_i) \leq \mu\varphi(\hat{f}_i)$. Now, the projections $p = \bar{e} - \sum_i \hat{e}_i$ and $q = \bar{f} - \sum_i \hat{f}_i$ are equivalent. If $p_1, p_2 \leq p$ with $p_1 \sim p_2$, then there exists $r \leq q$ such that $r \sim p_1$, giving

$$\begin{aligned}\varphi(p_1) &\leq \mu\varphi(r) \\ &\leq (1 + \varepsilon)\varphi(p_2).\end{aligned}$$

Since φ is bounded, and every positive operator can be approximated uniformly by the span of its spectral projections, it follows that

$$\varphi(ux^*xu) \leq (1 + \varepsilon)\varphi(x^*x)$$

for each $x \in pMp$ and unitary $u \in U(pMp)$. If we take the polar decomposition $x^* = v|x^*|$, then since pMp is finite, v can be extended to a unitary u^* , so $\varphi(xx^*) \leq (1 + \varepsilon)\varphi(x^*x)$. $\square$

Lemma: Let M be a finite von Neumann algebra, and let $\varepsilon > 0$. Then, there is a normal center-valued state φ such that for all $x \in M$, we have

$$\varphi(xx^*) \leq (1 + \varepsilon)\varphi(x^*x).$$

Proof. We only need to show the existence of such a state on Mz for some $z \in P(Z(M))$, as then we may use maximality to resolve this argument.

From above, we know that there exists a normal center-valued state ψ , and there exists some nonzero $p \in P(M)$ such that $\psi(xx^*) \leq (1 + \varepsilon)\psi(x^*x)$ for all $x \in pMp$. We may assume that p is monic.

Let $\{p_1, \dots, p_n\}$ be a finite family of pairwise orthogonal projections with $p_i \sim p$ and $z_0 = \sum_{i=1}^n p_i$, with $z_0 \in Z(M)$. Let $v_i \in M$ be such that $v_i^*v_i = p_i$ and $v_iv_i^* = p$. For any $x \in Mz_0$, define

$$\varphi_0(x) = \sum_{i=1}^n \psi(v_ixv_i^*).$$

Then, we have

$$\begin{aligned}0 &\leq \varphi_0(xx^*) \\ &= \varphi_0(xz_0x^*) \\ &= \sum_{j=1}^n \varphi_0(xp_jx^*) \\ &= \sum_{i,j=1}^n \psi(v_ixv_j^*v_jx^*v_i^*) \\ &\leq (1 + \varepsilon) \sum_{i,j=1}^n \psi(v_jx^*v_i^*v_ixv_j^*) \\ &= (1 + \varepsilon) \sum_{i=1}^n \varphi_i(x^*p_ix) \\ &= (1 + \varepsilon)\varphi_0(x^*x).\end{aligned}$$

For any $\bar{z} \in Z(M)z_0$ and $x \in Mz_0$, the fact that ψ is center-valued gives

$$\varphi_0(\bar{z}x) = \sum_{i=1}^n \psi(v_i\bar{z}xv_i^*)$$

$$= \bar{z}\varphi_0(x).$$

In general, though, we may not have $\varphi_0(z_0) = z_0$. Yet, since $\varphi_0(z_0) > 0$, we may take a spectral projection $z = \chi_{[\varepsilon, \infty)}(\varphi_0(z_0))$, so $0 \neq z = y\varphi_0(z_0)$ for some $y \geq 0$ and $y \in Z(M)$. Setting $\varphi(x) = y\varphi_0(x)$ gives us our desired result. $\square$

Theorem: A von Neumann algebra M is finite if and only if there exists a faithful normal center-valued trace.

Proof. If τ is any center-valued trace on M , and p is any monic projection, with $z = \sum_{i=1}^n p_i$ and each $p_i \sim p$, then

$$\begin{aligned} z &= \tau\left(\sum_{i=1}^n p_i\right) \\ &= n\tau(p), \end{aligned}$$

so that $\tau(p) > 0$. Then, $\tau(p) > 0$ for any nonzero projection, so $\tau(x) > 0$ for any nonzero positive operator $x \in M$, meaning τ is faithful.

Suppose there exists such a trace. If $v \in M$ is an isometry, then $1 = \tau(v^*v) = \tau(vv^*)$, so $\tau(1 - vv^*) = 1$, so $vv^* = 1$, meaning M is finite.

Conversely, suppose M is finite. Then, if $(a_n)_n$ is a sequence of real numbers that converge to 1, then there exists a sequence of normal center-valued states τ_n such that $\tau_n(xx^*) \leq a_n\tau_n(x^*x)$ for each $n \in \mathbb{N}$ and $x \in M$. We claim that if $1 \leq m < n$, then $a_m^2\tau_m - \tau_n$ is a positive linear map. To see this, it is enough to consider projections; since the map is normal, it is enough to consider monic projections. Let $p \sim p_1 \sim \dots \sim p_k$ be nonzero projections such that $z = \sum_{i=1}^k p_i$ is a central projection. Then, $\tau_n(p) \leq a_n\tau_n(p_i)$ and $\tau_m \leq a_m\tau_m(p_i)$ for each $1 \leq i \leq k$. Thus,

$$\begin{aligned} k\tau_n(p) &\leq a_n \sum_{i=1}^k \tau_n(p_i) \\ &= a_n\tau_n(z) \\ &= a_n z \\ &= a_n\tau_m(z) \\ &= a_n \sum_{i=1}^k \tau_m(p_i) \\ &\leq ka_n a_m \tau_m(p) \\ &\leq ka_m^2 \tau_m(p). \end{aligned}$$

Thus, $\|a_m^2\tau_m - \tau_n\| \leq a_m^2 - 1$, so there is a bounded linear map τ such that $\|\tau - \tau_m\| \rightarrow 0$.

We have $0 \leq \tau(xx^*) \leq \tau(x^*x)$, but since this holds for all $x \in M$, it follows that we must have equality $\tau(x^*x) = \tau(xx^*)$. Using the polar decomposition of x gives that $\tau(u^*yu) = \tau(y)$ for all invertible $y \geq 0$. Taking linear combinations shows that $\tau(y)$ is invariant under unitary conjugation, so τ is a trace.

Furthermore, $\tau|_{Z(M)} = \text{id}$, and $\tau(zx) = z\tau(x)$ for all $z \in Z(M)$ and $x \in M$, so we only need to verify that τ is normal. Letting $\varphi \in M_*$ be any normal linear functional, then

$$\|\varphi \circ \tau - \varphi \circ \tau_m\| \leq \|\varphi\| \|\tau - \tau_m\|,$$

so $\varphi \circ \tau$ is in M_* since M_* is closed, meaning τ is normal. $\square$

Proposition: Let M be a finite von Neumann algebra with center-valued trace τ . If $p, q \in P(M)$, then $p \leq q$ if and only if $\tau(p) \leq \tau(q)$.

Proof. Let $p = v^*v$ and $vv^* \leq q$. Then, $\tau(p) = \tau(vv^*) \leq \tau(q)$ as τ is positive.

Conversely, if $\tau(p) \leq \tau(q)$, then by the comparison theorem, we have $z \in Z(P(M))$ such that $pz \leq qz$ and $(1-z)q \leq (1-z)p$. If $v^*v = (1-z)q$ and $vv^* \leq (1-z)p$, then

$$\begin{aligned} \tau(vv^*) &\leq \tau((1-z)p) \\ &= (1-z)\tau(p) \\ &\leq (1-z)\tau(q) \\ &= \tau(vv^*), \end{aligned}$$

so since τ is faithful, $vv^* = (1-z)p$, so $(1-z)q \sim (1-z)p$ and $p \leq q$. $\square$

Now, we start discussing the special case of II_1 factors.

Proposition: Let M be a II_1 factor with trace τ . Then, $p \mapsto \tau(p)$ induces a bijection from the equivalence classes of projections in M onto $[0, 1]$.

Proof. We only need to show that for all $t \in (0, 1)$, there is $p \in P(M)$ with $\tau(p) = t$. If $t = \sum_k 2^{-n_k}$ is the dyadic expansion of t , then we may use the mutually orthogonal projections p_{n_k} indexed by dyadic rationals that have $\tau(p_{n_k}) = 2^{-n_k}$ discussed at the beginning of this section to construct $p = \sum_k p_{n_k}$. Since τ is normal, it is continuous with respect to norm-bounded increasing nets of projections, so $\tau(p) = t$. $\square$

For all $n \in \mathbb{N}$, $\mathbb{M}_n(M)$ is also a II_1 factor, with its trace given by $\tau_n\left(\left(x_{ij}\right)_{i,j}\right) = \frac{1}{n} \sum_{i=1}^n \tau(x_{ii})$.

If $0 < t \leq n$, then for any projections $p, q \in P(\mathbb{M}_n(M))$ with $\tau_n(p) = \tau_n(q) = t/n$, there is a unitary $u \in U(\mathbb{M}_n(M))$ such that $upMp^* = qMq$. In particular, up to isomorphism, the corner pMp only depends on t . Furthermore, this is independent of n since any two matrix algebras over M can be embedded into a larger common matrix algebra. The *amplification* of M with respect to this parameter t is the factor $M^t := pMp$, where $\tau(p) = t$.

We have that $(M^t)^s \cong M^{ts}$ for all $t, s > 0$. The fundamental group^{II} of M is the set of all $t \in \mathbb{R}_{>0}$ with $M^t \cong M$. We have that the fundamental group is an isomorphism class invariant of M .

Conditional Expectations

This subsection is also in the [completely positive maps](#) notes.

Definition: If A is a unital C^* -algebra, and $B \subseteq A$ is a unital C^* -subalgebra, then a conditional expectation from A to B is a unital completely positive map $E: A \rightarrow B$ with $E|_B = \text{id}$.

From [Pau02, Corollary 3.19], every conditional expectation is bimodular — that is, $E(axb) = aE(x)b$ for all $a, b \in B$ and $x \in A$.

It turns out that the stated properties are defining for conditional expectations.

Theorem: Let A be a unital C^* -algebra, $B \subseteq A$ a unital C^* -subalgebra, and $E: A \rightarrow B$ with $E|_B = \text{id}$ and $\|E\| \leq 1$. Then, E is a conditional expectation.

Proof. Start by considering the case that A is a von Neumann algebra and B is a von Neumann subalgebra.

Then, for any $p \in P(B)$ and $x \in A$, we have $(1-p)E(px) = E((1-p)E(px))$, so for all $t > 0$,

$$\begin{aligned} (1+t)^2 \|pE((1-p)x)\|^2 &= \|pE((1-p)x) + tpE((1-p)x)\|^2 \\ &\leq \|(1-p)x + tpE((1-p)x)\|^2 \\ &\leq \|(1-p)x\|^2 + t^2 \|pE((1-p)x)\|^2, \end{aligned}$$

so for all $t > 0$, we have $(1+2t)\|pE((1-p)x)\|^2 \leq \|(1-p)x\|^2$, so that $pE((1-p)x) = 0$. Since both p and $1-p$ are in B , we have that $pE(x) = pE(px) = E(px)$. Since the span of the projections is norm-dense in B , we have $E(yx) = yE(x)$ for all $y \in B$ and all $x \in A$. Therefore, we have that E is B -bimodular.

^{II}No relation to the topological one.

Since E is a unital contraction, we have that E is positive. If $(a_{ij})_{i,j} \in \mathbb{M}_n(A)_+$ and $x_1, \dots, x_n \in B$, then we have

$$\sum_{i,j=1}^n x_i^* E(a_{ij}) x_j = E \left(\sum_{i,j=1}^n x_i^* a_{ij} x_j \right) \geq 0,$$

so $E^{(n)}((a_{ij})_{i,j}) \geq 0$, meaning E is completely positive.

In the general case, we embed A into its double dual, A^{**} , which is a von Neumann algebra, and similarly for B , and use the fact that $E^{**}: A^{**} \rightarrow B^{**}$ is equal to E on the subalgebras A and B . $\square$

Theorem: Let M be a finite von Neumann algebra with trace τ , and let $N \subseteq M$ be a von Neumann subalgebra. Then, there exists a unique normal conditional expectation $E: M \rightarrow N$ with $\tau \circ E = \tau$.

Proof. Let $e_N \in B(L_2(M, \tau))$ be the projection onto $L_2(N, \tau) \subseteq L_2(M, \tau)$. Let J be the conjugation operator on $L_2(M, \tau)$, which we can also view as the conjugation operator on $L_2(N, \tau)$.

We have $N' \cap B(L_2(N, \tau))$ from a basic result on the standard representation (see [orbit equivalence notes](#)). Since $L_2(N, \tau)$ is invariant under the operation JNJ , we must have $e_N(JyJ) = (JyJ)e_N$ for all $y \in N$.

For any $x \in M$ and $y \in N$, we get

$$\begin{aligned} e_N x e_N JyJ &= e_N x JyJ \\ &= e_N x JyJ x e_N \\ &= JyJ e_N x e_N, \end{aligned}$$

so $e_N x e_N \in (JNJ)' = N$, and we may denote this operator by $E(x)$. We have that $E: M \rightarrow N$ is normal, unital, completely positive, and $E|_N = \text{id}$, so E is a normal conditional expectation. Furthermore, for any $x \in M$, we have

$$\begin{aligned} \tau(E(x)) &= \langle e_N x e_N 1_\tau, 1_\tau \rangle \\ &= \langle x 1_\tau, 1_\tau \rangle \\ &= \tau(x), \end{aligned}$$

where 1_τ denotes the cyclic vector on the GNS representation.

If $\hat{E}$ were another trace-preserving conditional expectation, then for any $x \in M$ and $y \in N$, we would have

$$\begin{aligned} \tau(\hat{E}(x)y) &= \tau(\hat{E}(xy)) \\ &= \tau(xy) \\ &= \tau(E(xy)) \\ &= \tau(E(x)y), \end{aligned}$$

meaning that $\hat{E} = E$ as x is arbitrary and τ is faithful. $\square$

Semifinite Traces

Similar to how, over a compact Hausdorff space K , states on $C(K)$ correspond to Radon probability measures on K . The notion of weight helps generalize the notion of not-necessarily-finite positive Radon measures.

Definition: A *weight* on a C^* -algebra A is a map $\varphi: A_+ \rightarrow [0, \infty]$ satisfying

$$\begin{aligned} \varphi(x+y) &= \varphi(x) + \varphi(y) \\ \varphi(\lambda x) &= \lambda \varphi(x) \end{aligned}$$

$$\varphi(0) = 0$$

for all $x, y \in A_+$ and $\lambda > 0$.

The weight is faithful if $\varphi(x) \neq 0$ for all $x \neq 0$ and φ is tracial if $\varphi(x^*x) = \varphi(xx^*)$ for all $x \in A$.

If A is a von Neumann algebra, the weight φ is semifinite if the set

$$\mathfrak{p}_\varphi = \{x \in A_+ \mid \varphi(x) < \infty\}$$

generates A as a von Neumann algebra. We say φ is normal if it is a normal map in A_+ .

Proposition: Let φ be a weight on a C^* -algebra A . Then,

- (i) $\mathfrak{n}_\varphi = \{x \in A \mid x^*x \in \mathfrak{p}_\varphi\}$ is a left ideal of A ;
- (ii) $\mathfrak{m}_\varphi = \left\{ \sum_{i=1}^n y_i^* x_i \mid x_1, \dots, x_n, y_1, \dots, y_n \in \mathfrak{n}_\varphi \right\}$ is a $*$ -subalgebra of $\mathfrak{n}_\varphi$ that is the span of $\mathfrak{p}_\varphi$.

Proof. Since

$$\begin{aligned} (x+y)^*(x+y) &\leq (x+y)^*(x+y) + (x-y)^*(x-y) \\ &= 2(x^*x + y^*y), \end{aligned}$$

we have that $\mathfrak{n}_\varphi$ is a linear subspace. It is a left ideal since $(ax)^*(ax) \leq \|a\|^2 x^*x$.

Since $\mathfrak{n}_\varphi$ is a linear subspace, we have that $\mathfrak{m}_\varphi$ is a $*$ -subalgebra. We have that $\mathfrak{m}_\varphi$ contains $\mathfrak{p}_\varphi$, and since

$$4y^*x = \sum_{k=0}^3 i^k (x + i^k y)^* (x + i^k y),$$

that $\mathfrak{m}_\varphi$ is equal to the span of $\mathfrak{p}_\varphi$ follows. □

We call $\mathfrak{m}_\varphi$ the domain of definition for φ .

If $\varphi(1) < \infty$, then we can extend φ to all of A , and upon rescaling, construct a state.

Fixing a weight φ on a C^* -algebra A , then the set

$$N_\varphi = \{x \in A \mid \varphi(x^*x) = 0\}$$

is a left ideal of A that is contained in $\mathfrak{n}_\varphi$. The quotient space $\mathfrak{n}_\varphi/N_\varphi$ with quotient map $\eta_\varphi: \mathfrak{n}_\varphi \rightarrow \mathfrak{n}_\varphi/N_\varphi$ allows us to define a sesquilinear form on $\mathfrak{n}_\varphi/N_\varphi$ by

$$\langle \eta_\varphi(x), \eta_\varphi(y) \rangle = \varphi(y^*x).$$

The space $L_2(A, \varphi)$ is the Hilbert space completion with respect to this form, and the inequality $(ax)^*(ax) \leq \|a\|^2 x^*x$ extends to a representation $\pi_\varphi: A \rightarrow B(L_2(A, \varphi))$.

This is essentially a version of the GNS construction with respect to a semifinite weight.

Theorem: Let M be a semifinite factor. Then, there exists a unique up to scalar multiplication faithful normal semifinite tracial weight $\text{Tr}: M_+ \rightarrow [0, \infty]$. Furthermore, $p \in P(M)$ is finite if and only if $\text{Tr}(p) < \infty$.

Proof. Any finite trace is semifinite, so we only need to consider the case when M is properly infinite.

Let $p_0 \in M$ be a nonzero finite projection. Then, there exists an infinite family $\{p_i\}_{i \in I}$ with $\sum_{i \in I} p_i = 1$ and $p_0 \sim p_i$ for each $i \in I$. Let v_i be the partial isometries implementing this equivalence, with $v_i^* v_i = p_0$ and $v_i v_i^* = p_i$; let τ_0 be the unique tracial state on $p_0 M p_0$.

Define $\text{Tr}: M_+ \rightarrow [0, \infty]$ by

$$\text{Tr}(x) = \sum_{i \in I} \tau_0(v_i^* x v_i).$$

Then, Tr is a normal weight on M , with normality emerging from the fact that $x \mapsto \tau_0(v_i^* x v_i)$ is a normal map for each i .

Now, we have $v_i^* M v_i \subseteq \mathfrak{m}_{\text{Tr}}$ for each $i \in I$, meaning that $\mathfrak{m}_{\text{Tr}}$ is weakly dense in M , so Tr is semifinite.

For each $x \in M$, we use Fubini's Theorem to find

$$\begin{aligned} \text{Tr}(xx^*) &= \sum_{i,j \in I} \tau_0(v_i^* x v_j v_j^* x^* v_i) \\ &= \sum_{i,j \in I} \tau_0(v_j x^* v_i v_i^* x v_j^*) \\ &= \text{Tr}(x^* x). \end{aligned}$$

Now, if $p \in P(M)$ is finite, then so is the supremum $p_0 \vee p$. There is a nonzero monic subprojection $q \leq p_0$ in $(p_0 \vee p)M(p_0 \vee p)$ as this is a finite corner. We must have $0 < \text{Tr}(q)$, and $\text{Tr}(p_0 \vee p) < \infty$, so Tr is faithful and finite on finite projections.

Conversely, if p is infinite, then there is a subprojection $q \leq p$ such that $q \sim p$ and $q \sim p - q$, meaning that $\text{Tr}(p) = \text{Tr}(p - q) = \text{Tr}(q)$, so $\text{Tr}(p) = 2 \text{Tr}(p)$. Since Tr is faithful, $\text{Tr}(p) = \infty$.

We now show uniqueness. Let M be a semifinite factor and ω a normal semifinite tracial weight on M . Since ω is semifinite, for each $x \in M_+$ with $x \neq 0$, there is $y \in \mathfrak{n}_\omega$ such that $x^{1/2}y \neq 0$. Therefore, $x^{1/2}y y^* x^{1/2} \in \mathfrak{m}_\omega$, so $\omega(x^{1/2}y y^* x^{1/2}) < \infty$. If q is some nonzero spectral projection of $x^{1/2}y y^* x^{1/2}$, then there exists some $c > 0$ such that $cq \leq x^{1/2}y y^* x^{1/2} \leq \|y\|^2 x$.

From this, it follows that q is a finite projection, since $\omega(q) < \infty$, so $\omega(q) \neq 0$. In particular, this means that $\omega(x) \neq 0$, so ω is faithful. By a similar argument as with the case of Tr , it follows that $\omega(p)$ is finite if and only if p is finite. In particular, this means that $\omega|_{pMp}$ defines a tracial positive linear functional, so it is a scalar multiple of the trace on pMp . Since M is semifinite, every projection is an increasing limit of finite projections, so Tr is unique up to a scalar multiple. $\square$

Proposition: Let M be a countably decomposable semifinite factor. If $p, q \in P(M)$, then $p \leq q$ if and only if $\text{Tr}(p) \leq \text{Tr}(q)$.

Proof. If p and q are finite, then restricting Tr to the finite corner $(p \vee q)M(p \vee q)$ gives us a scalar multiple of the trace on $(p \vee q)M(p \vee q)$, so the conclusion follows from the case of finite von Neumann algebras.

Meanwhile, we have $\text{Tr}(p) = \infty$ if and only if p is not finite, so this follows from the fact that any two infinite projections are equivalent in a semifinite factor. $\square$

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