

Some of these notes were originally written by my advisor, Ben Hayes.

Overviewing Amenability in von Neumann Algebras

This material is taken from the [completely positive maps](#) notes.

Recall that a discrete group G is called amenable if it admits $m \in \ell_\infty(G)^*$ such that $m(\lambda_s f) = m(f)$, where $\lambda_s f(t) = f(s^{-1}t)$. There are a variety of other equivalent notions, such as the existence of a net $(\xi_i)_i \in \ell_1(G)$ satisfying

$$\|\lambda_s \xi_i - \xi_i\|_{\ell_1} \rightarrow 0$$

for all $s \in G$ (approximate means), or if $\lambda: G \rightarrow U(\ell_2(G))$ be the left regular representation, we have some net of unit vectors $(\xi_i)_i \subseteq S_{\ell_2(G)}$ satisfying

$$\|\lambda(s)\xi_i - \xi_i\|_{\ell_2} \rightarrow 0,$$

known as almost invariant vectors. The existence of almost invariant vectors is furthermore equivalent to the existence of a net $(F_i)_{i \in I}$ of finite subsets of G whose union is equal to G , with the net satisfying

$$\frac{|sF_i \Delta F_i|}{|F_i|} \rightarrow 0,$$

known as a Følner net (or sequence).

Most all of these have analogous ideas in von Neumann algebras. We will start by discussing an analogue of an invariant mean on a von Neumann algebra, known as a hypertrace.

Let (M, τ) be a von Neumann algebra, and let $L_2(M)$ denote the standard representation of M , which is the GNS representation of M with respect to the faithful state τ . We observe that $L_2(M)$ is equipped with an M - M -bimodule structure.

We start by introducing a way to characterize amenability of a group via its group von Neumann algebra.

Proposition: Let G be a group, and let $M = L(G)$ be its group von Neumann algebra. Then, G is amenable if and only if there exists a conditional expectation E from $B(L_2(M))$ onto M .

Proof. Suppose such an E exists. If $f \in \ell_\infty(G)$ is some function, let M_f be the multiplication operator on $\ell_2(G)$ corresponding to f . Set $m(f) = \tau(E(M_f))$, where τ is the canonical trace on M . Since $u_s M_f u_s^* = M_{\lambda_s f}$ for every $s \in G$, we have that m is left invariant.

Now, suppose G is amenable, and let m be a mean on $\ell_\infty(G)$. Given $\xi, \eta \in \ell_2(G)$, and $T \in B(L_2(M))$, let

$$f_{\xi, \eta}^T(s) = \langle \xi, \rho(s) T \rho(s^{-1}) \eta \rangle,$$

where ρ is the right regular representation of G . Then, $f_{\xi, \eta}^T$ is a bounded function on G satisfying

$$|f_{\xi, \eta}^T(s)| \leq \|T\| \|\xi\| \|\eta\|,$$

which allows us to define a continuous sesquilinear form

$$(\xi, \eta) = m(f_{\xi, \eta}^T).$$

By the structure of sesquilinear form, we have a unique operator $E(T)$ satisfying

$$\langle \xi, E(T) \eta \rangle = m(f_{\xi, \eta}^T).$$

Since m is invariant under the left regular action, it is invariant under the right regular action, so that

$$\rho(s) E(T) \rho(s^{-1}) = E(T),$$

meaning that $E(T)$ commutes with $\rho(G)$, whence $E(T) \in L(G)$. Furthermore, $E(T)$ is a conditional expectation as it is equal to the identity on $L(G)$ and has norm less than or equal to 1. $\square$

This property motivates the following definition.

Definition: We say a von Neumann algebra is *amenable* (or *injective*) if M can be faithfully represented as a von Neumann subalgebra of some $B(H)$ such that there exists a conditional expectation $E: B(H) \rightarrow M$.

Proposition: Let M be a von Neumann algebra. The following are equivalent:

- (i) M is injective;
- (ii) for every inclusion $A \subseteq B$ of unital C^* -subalgebras, any unital completely positive map $\phi: A \rightarrow M$ extends to a completely positive map from B to M ;
- (iii) for any $B(H)$ that contains M as a von Neumann subalgebra, there is a conditional expectation from $B(H)$ onto M .

Proof. To show (i) implies (ii), assume M is a von Neumann subalgebra of $B(H)$, and let $E: B(H) \rightarrow M$ be a conditional expectation. Extend ϕ to a completely positive map $\tilde{\phi}: B \rightarrow B(H)$ by using Arveson's extension theorem. Then, $E \circ \tilde{\phi}: B \rightarrow M$ is a completely positive extension of ϕ .

For (ii) implies (iii), if $M \subseteq B(H)$, then apply (ii) with $A = M$, $B = B(H)$, and the identity map id_M . Then, there exists a completely positive map $\phi: B(H) \rightarrow M$ whose restriction to M is id_M , which is necessarily a conditional expectation. $\square$

Example: The hyperfinite II_1 factor, R , is amenable. Recall that the hyperfinite II_1 factor is defined to be the SOT-closure of the inductive limit of $Q_n = \mathbb{M}_{2^n}(\mathbb{C})$, where we equip $Q_n \rightarrow Q_{n+1}$ with the block-diagonal embedding,

$$A \mapsto \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}.$$

Let J be the conjugation operator on $L_2(R)$. For any $n \geq 1$ and $T \in B(L_2(R))$, set

$$E_n(T) = J \left(\int_{U(2^n)} u J T J u^* d\mu(u) \right) J,$$

where μ denotes the Haar probability measure on the compact group $U(2^n)$, the group of unitary $2^n \times 2^n$ matrices. Then, $(E_n(T))_n$ is a norm-bounded sequence of operators in $B(L_2(R))$.

Take the limit of this sequence along a free ultrafilter, which allows for the existence of a unique bounded operator denoted $E(T)$ satisfying

$$\langle \xi, E(T)\eta \rangle = \lim_{\omega} \langle \xi, E_n(T)\eta \rangle$$

for all $\xi, \eta \in L_2(R)$. Observe that for any $n \geq n_0$, we have that $E_n(T) \in J(Q_{n_0})'J$, so that

$$\begin{aligned} E(T) &\in \bigcap_{n_0 \geq 1} J(Q_{n_0})'J \\ &= R, \end{aligned}$$

and $E(T) = T$ if $T \in R$. This gives that E is our desired conditional expectation.

Definition: Let (M, τ) be a tracial von Neumann algebra. A state ψ on $B(L_2(M, \tau))$ is called a *hypertrace* if $\psi(xT) = \psi(Tx)$ for every $x \in M$ and every $T \in B(L_2(M, \tau))$, with $\psi|_M = \tau$.

Proposition: Let (M, τ) be a tracial von Neumann algebra. Then, M is amenable if and only if it has a hypertrace.

Proof. If M is amenable, then since $M \subseteq B(L_2(M, \tau))$, we have a conditional expectation $E: B(L_2(M, \tau)) \rightarrow M$, so that $\tau \circ E$ is a hypertrace.

Conversely, if ψ is a hypertrace, then given $T \in B(L_2(M, \tau))_+$, we may define a positive linear functional φ on M by taking $\varphi(x) = \psi(Tx)$. For any $x \in M_+$, we then have

$$\begin{aligned} |\varphi(x)|^2 &= \left| \psi(x^{1/2} T x^{1/2}) \right|^2 \\ &\leq \psi(x^{1/2} T^2 x^{1/2}) \psi(x) \\ &\leq \|T\|^2 \psi(x)^2. \end{aligned}$$

Since $\psi|_M = \tau$, we have $\varphi(x) \leq \|T\| \tau(x)$ for any $x \in M_+$, and in particular, φ is normal. By the noncommutative Radon–Nikodym theorem, there is some $E(T) \in M_+$ such that for all $x \in M$,

$$\psi(Tx) = \tau(E(T)x).$$

We may extend E to yield a conditional expectation from $B(L_2(M, \tau)) \rightarrow M$. □

Approximation-Type Characterizations

Next, we will discuss some approximation-type properties (akin to the almost-invariant vectors characterization of amenability of G). For this, we must provide an overview of the trace-class and Hilbert–Schmidt operators on the space $L_2(M)$. This will allow us to use the Powers–Størmer inequality to prove that amenability of a von Neumann algebra is equivalent to the existence of almost central vectors in the coarse M - M -bimodule $L_2(M) \otimes L_2(M)$.

Definition: If H is a Hilbert space, then we set

$$\begin{aligned} S^1(H) &= \left\{ T \in K(H) \mid \text{Tr}(T) = \sum_{i \in I} \langle T|e_i, e_i \rangle < \infty \right\} \\ S^2(H) &= \{ T \in K(H) \mid \text{Tr}(T^*T) < \infty \}. \end{aligned}$$

We call $S^1(H)$ the trace-class operators, and $S^2(H)$ the Hilbert–Schmidt operators. The Hilbert–Schmidt operators admit an inner product and norm defined by

$$\begin{aligned} \langle A, B \rangle &= \text{Tr}(B^*A) \\ \|A\|_{\text{HS}} &= \text{Tr}(A^*A)^{1/2}. \end{aligned}$$

Some of the following properties can be shown:

- the value of $\text{Tr}(A)$ is independent of orthonormal basis;
- the space $S^2(H)$ is a $B(H)$ - $B(H)$ -bimodule, as $\|AB\|_{\text{HS}} \leq \|A\| \|B\|_{\text{HS}}$ and $\|BA\|_{\text{HS}} \leq \|A\| \|B\|_{\text{HS}}$, where $A \in B(H)$ and $B \in S^2(H)$;
- the trace is tracial, in that $\text{Tr}(AB) = \text{Tr}(BA)$;
- the value $\|A\|_{\text{Tr}} = \text{Tr}(|A|)$ defines a norm on $S^1(H)$;
- the space $S^1(H)$ is a $B(H)$ - $B(H)$ -bimodule;
- if $A, B \in S^2(H)$, then $\|AB\|_{\text{Tr}} \leq \|A\|_{\text{HS}} \|B\|_{\text{HS}}$.

We will focus on showing some of the properties for $\|\cdot\|_{\text{Tr}}$. This follows from the fact that if $A, B \in B(H)$, then

$$\begin{aligned} 0 &\leq (A - B)^*(A - B) \\ &= A^*A - A^*B - B^*A + B^*B \end{aligned}$$

$$= -2 \operatorname{Re}(B^*A) + |A|^2 + |B|^2,$$

whence

$$\operatorname{Re}(B^*A) \leq \frac{1}{2}(|A|^2 + |B|^2).$$

If $A \in S^1(H)$ is any trace-class operator and $B \in B(H)$ any operator where we have $1 = \|A\|_{\operatorname{Tr}} = \|B\|$, and we have partial isometries $AB = V|AB|$ and $A = W|A|$, then we observe that

$$\begin{aligned} \|AB\|_{\operatorname{Tr}} &= \operatorname{Tr}(|AB|) \\ &= \operatorname{Tr}(V^*AB) \\ &= \operatorname{Tr}(V^*W^*|A|^{1/2}|A|^{1/2}B) \\ &= \operatorname{Tr}\left(\left(B^*|A|^{1/2}\right)^* V^*W^*|A|^{1/2}\right) \\ &\leq \frac{1}{2}\left(\operatorname{Tr}\left(|A|^{1/2}BB^*|A|^{1/2}\right) + \operatorname{Tr}\left(|A|^{1/2}VV^*W^*|A|^{1/2}\right)\right) \\ &\leq \frac{1}{2}(\|B\| \|A\|_{\operatorname{Tr}} + \|A\|_{\operatorname{Tr}}) \\ &= 1, \end{aligned}$$

whence $\|AB\|_{\operatorname{Tr}} \leq \|A\|_{\operatorname{Tr}} \|B\|$, and similarly for BA .

Additionally, if $A + B = V|A + B|$, then

$$\|A + B\|_{\operatorname{Tr}} \leq \|A\|_{\operatorname{Tr}} + \|B\|_{\operatorname{Tr}}.$$

We know from some von Neumann algebra theory that the trace class operators are the predual of $B(H)$, and we may identify $S^2(H)$ with $H \otimes H^*$ by taking $\xi \otimes \eta^* \mapsto \Theta_{\xi, \eta}$, where

$$\Theta_{\xi, \eta}(\zeta) = \langle \zeta, \eta \rangle \xi.$$

Proposition (Powers–Størmer Inequality): If $A, B \in S^2(H)$ are positive, then

$$\begin{aligned} \|A - B\|_{\operatorname{HS}}^2 &\leq \|A^2 - B^2\|_{\operatorname{Tr}} \\ &\leq \|A + B\|_{\operatorname{HS}} \|A - B\|_{\operatorname{HS}}. \end{aligned}$$

Proof. We have

$$A^2 - B^2 = \frac{1}{2}((A + B)(A - B) + (A - B)(A + B)),$$

whence

$$\begin{aligned} \|A^2 - B^2\|_{\operatorname{Tr}} &\leq \frac{1}{2}(\|A + B\|_{\operatorname{HS}} \|A - B\|_{\operatorname{HS}} + \|A - B\|_{\operatorname{HS}} \|A + B\|_{\operatorname{HS}}) \\ &= \|A + B\|_{\operatorname{HS}} \|A - B\|_{\operatorname{HS}}. \end{aligned}$$

For the other inequality, we observe that if $x \in S^1(H)_+$ and $y \in B(H)_+$, then $\operatorname{Tr}(xy) = \operatorname{Tr}(y^{1/2}xy^{1/2}) \geq 0$.

Let $e = \chi_{[0, \infty)}(A - B)$, and let $e^\perp := 1 - e$. Then, we have

$$\begin{aligned} \operatorname{Tr}((A - B)^2) &= \operatorname{Tr}((A - B)(A - B)e) + \operatorname{Tr}((B - A)(B - A)e^\perp) \\ &\leq \operatorname{Tr}((A + B)(A - B)e) + \operatorname{Tr}((B + A)(B - A)e^\perp). \end{aligned}$$

Since e is a spectral projection of the operator $A - B$, it follows that e and $A - B$ commute. Therefore, we have

$$\operatorname{Tr}((A + B)(A - B)e) = \operatorname{Tr}((A + B)e(A - B))$$

$$= \text{Tr}((A - B)(A + B)e),$$

whence

$$\begin{aligned} \text{Tr}((A - B)^2) &\leq \text{Tr}((A^2 - B^2)e) + \text{Tr}((B^2 - A^2)e^\perp) \\ &\leq \text{Tr}(|A^2 - B^2|e) + \text{Tr}(|B^2 - A^2|e^\perp) \\ &= \text{Tr}(|A^2 - B^2|). \end{aligned}$$

□

The Powers–Størmer will allow us to establish the following equivalence, which provides an analogue to the almost invariant vectors/approximate mean conditions for amenability in groups but for injectivity of von Neumann algebras.

Theorem: Let M be a II_1 factor. The following are equivalent:

- (i) M has a hypertrace;
- (ii) there exists $(A_i)_i \in S^1(H)$ with $A_i \geq 0$ and $\text{Tr}(A_i) = 1$ such that $\| [x, A_i] \|_{\text{Tr}} \rightarrow 0$ for all $x \in M$;
- (iii) there exists $(\xi_i)_i \in L_2(M) \otimes L_2(M)$ such that $\| \xi_i \|_{\text{HS}} = 1$ and $\| [x, \xi_i] \|_{\text{HS}} \rightarrow 0$ for all $x \in M$.

Proof. If (ii) holds, then we realize $S^2(L_2(M))$ as $L_2(M) \otimes L_2(M)$, where we map $\xi \otimes \eta \in L_2(M) \otimes L_2(M)$ to $\Theta_{\xi, J\eta} \in S^2(L_2(M))$, where J is the canonical conjugation isometry. Setting $\xi_i = A_i^2$ and applying the Powers–Størmer inequality gives, for all unitaries $u \in U(M)$

$$\begin{aligned} \| u \xi_i u^* - \xi_i \|_{\text{HS}} &\leq \| u A_i^2 u^* - A_i^2 \|_{\text{Tr}} \\ &\rightarrow 0. \end{aligned}$$

Therefore, (iii) holds.

Now, if (iii) holds, we let $\varphi \in M^*$ be the linear functional such that $\langle \cdot, \xi_i \rangle \rightarrow \varphi$ weak*. Then, for all $x \in M$ and $y \in B(L_2(M))$, we have

$$|\varphi(xy) - \varphi(yx)| = \lim_{i \in I} |\langle xy \xi_i, \xi_i \rangle - \langle yx \xi_i, \xi_i \rangle|.$$

Since $\| x \xi_i - \xi_i x \|_{\text{HS}} \rightarrow 0$, we have

$$\begin{aligned} &= \lim_{i \in I} |\langle xy \xi_i, \xi_i \rangle - \langle y \xi_i x, \xi_i \rangle| \\ &= \lim_{i \in I} |\langle xy \xi_i, \xi_i \rangle - \langle y \xi_i, \xi_i x^* \rangle| \\ &= \lim_{i \in I} |\langle xy \xi_i, \xi_i \rangle - \langle y \xi_i, x^* \xi_i \rangle| \\ &= 0. \end{aligned}$$

Thus, φ is our desired hypertrace.

Now, suppose φ is a hypertrace.

We start by observing a general lemma. If $M \subseteq B(H)$ is a von Neumann algebra, then

$$S(M) = \underbrace{\{ \text{Tr}(\cdot A) \mid A \in S^1(H)_+, \text{Tr}(A) = 1 \}}_{=: K}^{w^*},$$

where we view $B(H) = (S^1(H))^*$. Observe that both sides are w^* -compact and convex, meaning that $S(M)$ containing K follows immediately from the fact that $\text{Tr}(A) = 1$. For inclusion, suppose $\varphi \in S(M) \setminus K$. It follows that there is some w^* -continuous linear functional $L: M^* \rightarrow \mathbb{C}$ and $\alpha < \beta \in \mathbb{R}$ such that $\text{Re}(L(\varphi)) \geq$

$\alpha > \beta \geq \operatorname{Re}(L(\psi))$ for all $\psi \in K$. Since $(M^*, w^*)^* = M$, there is some $a \in M$ such that $L(\rho) = \rho(a)$ for all $\rho \in M^*$. In particular, this means we may write

$$\begin{aligned} \operatorname{Re}(\varphi(a)) &\geq \alpha \\ &> \beta \\ &\geq \operatorname{Re}(\operatorname{Tr}(Aa)) \end{aligned}$$

for all $A \in S^1(H)_+$. In particular, this holds for the vector state, whence we may assume

$$\begin{aligned} \varphi(\operatorname{Re}(a)) &\geq \alpha \\ &> \beta \\ &\geq \langle \operatorname{Re}(a)\xi, \xi \rangle \end{aligned}$$

for all $\xi \in H$ with $\|\xi\| = 1$. Thus, without loss of generality, $a = a^*$. Taking the supremum over all satisfactory ξ gives

$$\begin{aligned} \sup_{\|\xi\|=1} \langle a\xi, \xi \rangle &= \|a_+\| \\ &\leq \beta \\ &< \alpha \\ &\leq \varphi(a) \\ &= \varphi(a_+) - \varphi(a_-) \\ &\leq \varphi(a_+) \\ &\leq \|a_+\|, \end{aligned}$$

which gives our desired contradiction.

In order to show (ii), it suffices to show that for all $x_1, \dots, x_n \in M$, we have

$$\begin{aligned} 0 &\in \overline{\left\{ \left([x_j, A] \right)_{j=1}^n \mid A \in S^1(H)_+, \operatorname{Tr}(A) = 1 \right\}}^{\|\cdot\|_{\operatorname{Tr}}} \\ &\subseteq S^1(H)^{(n)}. \end{aligned}$$

Since this is convex, the weak and norm closures coincide, where the weak topology is defined via the trace. By the lemma, if $\varphi: B(H) \rightarrow \mathbb{C}$ is a hypertrace for M , then there exists $(A_i)_{i \in I}$ in $S^1(H)_+$ with $\operatorname{Tr}(A_i) = 1$ for each i such that $\lim_i \operatorname{Tr}(\cdot A_i) = \varphi$.

It thus suffices to show that for all $B \in B(H)$, we have $\operatorname{Tr}([x_j, A_i]B) \rightarrow 0$ for each x_j . This follows from a computation, where we use traciality and the fact that the trace-class operators are a two-sided ideal, giving

$$\begin{aligned} \operatorname{Tr}([x_j, A_i]B) &= \operatorname{Tr}(x_j A_i B) - \operatorname{Tr}(A_i x_j B) \\ &= \operatorname{Tr}(B x_j A_i) - \operatorname{Tr}(x_j B A_i) \\ &\rightarrow \varphi(B x_j - x_j B) \\ &= 0, \end{aligned}$$

since φ is a hypertrace. Therefore, for all $j = 1, \dots, n$, we have $[x_j, A_i] \rightarrow 0$ weakly, giving our desired result. $\square$

Connes's Trick and Følner Projections

Thus far, we have shown an equivalence between amenability/injectivity, the existence of a hypertrace, and certain almost-invariant or almost-central vectors. The next step is to construct an analogue to the Følner sequences, and for this, we must use a noncommutative version of the classic Namioka trick.

It can be observed (and is a classic exercise of real analysis) that for any $s_1, s_2 \in \mathbb{R}$, we have

$$|s_1 - s_2| = \int_{-\infty}^{\infty} |\chi_{(t, \infty)}(s_1) - \chi_{(t, \infty)}(s_2)| dt.$$

Applying various bootstrapping measures, we have that if (X, μ) is a σ -finite measure space, then

$$\begin{aligned} \|f - g\|_{L_1} &= \int_{-\infty}^{\infty} \|\chi_{f^{-1}(t, \infty)} - \chi_{g^{-1}(t, \infty)}\|_{L_1} dt \\ \|f\|_{L_1} &= \int_{-\infty}^{\infty} \|\chi_{f^{-1}(t, \infty)}\|_{L_1} dt \end{aligned}$$

In particular, if we have $\|f - g\|_{L_1} < \varepsilon \|f\|_{L_1}$ for some $\varepsilon > 0$, then there is some $t_0 \in \mathbb{R}$ such that

$$\|\chi_{f^{-1}(t_0, \infty)} - \chi_{g^{-1}(t_0, \infty)}\|_{L_1} < \varepsilon \|\chi_{f^{-1}(t_0, \infty)}\|_{L_1}.$$

We will seek to obtain a noncommutative version of the Namioka trick. First, we must reduce computations of Hilbert–Schmidt norms to computations of L_2 norms of functions.

Proposition: Let H be a Hilbert space, and let x, y be two positive finite rank operators on H . There exists a positive Radon measure ν on $\mathbb{R}_+^2$ such that for every pair of Borel complex-valued functions on $\mathbb{R}_+$ with $f(0) = g(0) = 0$, we have

$$\|f(x) - g(y)\|_{\text{HS}}^2 = \int_{\mathbb{R}_+^2} |f(\alpha) - g(\beta)|^2 d\nu(\alpha, \beta).$$

Proof. Write

$$\begin{aligned} x &= \sum_{i=1}^m \lambda_i e_i \\ y &= \sum_{j=1}^n \mu_j f_j, \end{aligned}$$

where the λ_i and μ_j are the distinct strictly positive eigenvalues of x and y respectively, and the e_i and f_j are the corresponding spectral projections. Set

$$\begin{aligned} e_0 &= 1 - \sum_{i=1}^m e_i \\ f_0 &= 1 - \sum_{j=1}^n f_j, \end{aligned}$$

and let Tr denote the usual trace on $B(H)_+$.

Define

$$X = (\sigma(x) \times \sigma(y)) \setminus \{(0, 0)\},$$

where $\sigma(x)$ denotes the spectrum, and define a measure μ on X by setting

$$\nu(\{(\lambda_i, \mu_j)\}) = \text{Tr}(e_i f_j)$$

$$\begin{aligned} v(\{(\lambda_i, 0)\}) &= \text{Tr}(e_i f_0) \\ v(\{(0, \mu_j)\}) &= \text{Tr}(e_0 f_j). \end{aligned}$$

If we have two real-valued Borel functions on $\mathbb{R}_+$ with $f(0) = g(0) = 0$, then we have

$$\begin{aligned} f(x) &= \sum_{i=1}^m f(\lambda_i) e_i \\ g(y) &= \sum_{j=1}^n g(\mu_j) f_j \end{aligned}$$

are still finite-rank operators, and

$$\begin{aligned} \text{Tr}(f(x)g(y)) &= \int_{\mathbb{R}_+^2} f(\alpha)g(\beta) d\nu(\alpha, \beta) \\ \text{Tr}(f(x)^2) &= \int_{\mathbb{R}_+^2} f(\alpha)^2 d\nu(\alpha, \beta) \\ \text{Tr}(g(y)^2) &= \int_{\mathbb{R}_+^2} g(\beta)^2 d\nu(\alpha, \beta), \end{aligned}$$

so that

$$\|f(x) - g(y)\|_{\text{HS}}^2 = \int_{\mathbb{R}_+^2} |f(\alpha) - g(\beta)|^2 d\nu(\alpha, \beta).$$

□

Theorem (Connes's Trick): Let $x_1, \dots, x_n$ be positive finite-rank elements of $B(H)$, and $0 < \varepsilon < 1$ such that $\|x_i - x_1\|_{\text{HS}} < \varepsilon \|x_1\|_{\text{HS}}$. Then, there is $t_0 > 0$ such that

$$\|\chi_{(t_0, \infty)}(x_i) - \chi_{(t_0, \infty)}(x_1)\|_{\text{HS}} < (3n\varepsilon)^{1/2} \|\chi_{(t_0, \infty)}(x_1)\|_{\text{HS}}.$$

Proof. Applying the earlier lemma with $f = g = \chi_{(t^{1/2}, \infty)}$ gives

$$\|\chi_{(t^{1/2}, \infty)}(x) - \chi_{(t^{1/2}, \infty)}(y)\|_{\text{HS}}^2 = \int |\chi_{(t^{1/2}, \infty)}(\alpha) - \chi_{(t^{1/2}, \infty)}(\beta)|^2 d\nu(\alpha, \beta).$$

Integrating with respect to t , using Fubini's Theorem, and Hölder's Inequality respectively gives

$$\begin{aligned} \int_0^\infty \|\chi_{(t^{1/2}, \infty)}(x) - \chi_{(t^{1/2}, \infty)}(y)\|_{\text{HS}}^2 dt &= \int_{\mathbb{R}_+^2} \left(\int_0^\infty |\chi_{(t^{1/2}, \infty)}(\alpha) - \chi_{(t^{1/2}, \infty)}(\beta)|^2 dt \right) d\nu(\alpha, \beta) \\ &= \int_{\mathbb{R}_+^2} |\alpha^2 - \beta^2|^2 d\nu(\alpha, \beta) \\ &\leq \left(\int_{\mathbb{R}_+^2} |\alpha - \beta|^2 d\nu(\alpha, \beta) \right)^{1/2} \left(\int_{\mathbb{R}_+^2} |\alpha + \beta|^2 d\nu(\alpha, \beta) \right) \\ &= \|x - y\|_{\text{HS}} \|x + y\|_{\text{HS}} \end{aligned}$$

For any $\varepsilon < 1$, the triangle inequality gives that

$$\|x_j + x_1\|_{\text{HS}} \leq 3\|x_1\|_{\text{HS}},$$

so we have

$$\int_0^\infty \sum_{i=1}^n \|\chi_{(t^{1/2}, \infty)}(x_j) - \chi_{(t^{1/2}, \infty)}(x_1)\|_{\text{HS}}^2 dt < 3n\varepsilon \|x_1\|_{\text{HS}}^2$$

$$= 3n\varepsilon \int_0^\infty \left\| \chi_{(t^{1/2}, \infty)}(x_1) \right\|_{\text{HS}}^2 dt.$$

Hence, the Namioka trick gives some $t_0 > 0$ with

$$\sum_{i=1}^n \left\| \chi_{(t_0, \infty)}(x_i) - \chi_{(t_0, \infty)}(x_1) \right\|_{\text{HS}}^2 < 3n\varepsilon \left\| \chi_{(t_0, \infty)}(x_1) \right\|_{\text{HS}}^2.$$

□

The theorem applies in any semifinite von Neumann algebra (M, Tr) instead of just $(B(H), \text{Tr})$ upon using one's preferred faithful representation.

Proposition: Let (M, τ) be a tracial von Neumann algebra, and let $\xi, \eta \in L_2(M)_+$. There exists a positive Radon measure ν on $\mathbb{R}_+^2$ such that for all Borel functions f, g on $\mathbb{R}_+$ with $f(0) = g(0) = 0$ and $f(\xi), g(\eta) \in L_2(M)$, the maps $(\alpha, \beta) \mapsto f(\alpha)$ and $(\alpha, \beta) \mapsto g(\beta)$ are square-integrable, with

$$\left\| f(\xi) - g(\eta) \right\|_{L_2(M)}^2 = \int_{\mathbb{R}_+^2} |f(\alpha) - g(\beta)|^2 d\nu(\alpha, \beta).$$

Theorem: Let (M, τ) be a tracial von Neumann algebra, and let $\xi_1, \dots, \xi_n \in L_2(M)_+$. Let $0 < \varepsilon < 1$ be such that

$$\left\| \xi_i - \xi_1 \right\|_{L_2(M)} < \varepsilon \left\| \xi_1 \right\|_{L_2(M)}.$$

Then, there is $t_0 > 0$ such that

$$\left\| \chi_{(t_0, \infty)}(\xi_i) - \chi_{(t_0, \infty)}(\xi_1) \right\|_{L_2(M)} < (3n\varepsilon)^{1/2} \left\| \chi_{(t_0, \infty)}(\xi_1) \right\|_{L_2(M)}.$$

The main benefit we will find from the Connes trick is the existence of Følner projections.

Corollary: If M is amenable, then there exist projections $p_i \in B(L_2(M))$ with $0 < \text{Tr}(p_i) < \infty$ and

$$\frac{\left\| [x, p_i] \right\|_{\text{Tr}}}{\text{Tr}(p_i)} \rightarrow 0$$

and

$$\frac{\text{Tr}(xp_i)}{\text{Tr}(p_i)} \rightarrow \tau(x)$$

for all $x \in M$, where Tr denotes the trace on $B(L_2(M))$.

Proof. Given $\varepsilon > 0$ and some subset $F \subseteq U(M)$ finite, there exists $A \in S^2(H)$ with $\|A\|_{\text{HS}} = 1$ and $\sum_{u \in F} \|uAu^* - A\|_{\text{HS}}^2 < \varepsilon$.

This gives

$$\begin{aligned} \int_0^\infty \sum_{u \in F} \left\| u \chi_{(t^{1/2}, \infty)}(A) u^* - \chi_{(t^{1/2}, \infty)}(A) \right\|_{\text{HS}}^2 dt &\leq 2 \sum_{u \in F} \|uAu^* - A\|_{\text{HS}}^2 \\ &< 2\varepsilon^2 \\ &= 2\varepsilon^2 \int_0^\infty \left\| \chi_{(t^{1/2}, \infty)}(A) \right\| dt, \end{aligned}$$

so there exists t_0 such that $p = \chi_{(t_0^{1/2}, \infty)}(A)$ has

$$\sum_{u \in F} \|upu^* - p\|_{\text{HS}}^2 < 2\varepsilon^2 \text{Tr}(p).$$

Therefore, we have

$$\| [u, p] \|_{\text{Tr}} \leq 2\epsilon^2 \text{Tr}(p),$$

which gives our desired net.

Moreover, if we have

$$\frac{\text{Tr}(\cdot p_i)}{\text{Tr}(p_i)} \rightarrow \psi$$

for some $\psi \in M^*$, we have that ψ is a trace, meaning that ψ is necessarily the trace. $\square$

The Local Approximation Property

Definition: Let (M, τ) be a finite von Neumann algebra. We say (M, τ) has the *local approximation property* if, for all $\epsilon > 0$, there exists $e \in P(M)$ such that for all finite $F \subseteq B_M$, there exists $q \in P(M)$ with $q \leq e$ and a finite-dimensional matrix algebra Q in qMq such that

- (i) $\| [q, eve] \|_{\tau} < \epsilon \|q\|_{\tau}$ for all $v \in F$;
- (ii) $\| qvq - E(qvq) \|_{\tau} < \epsilon \|q\|_{\tau}$, where $E: qMq \rightarrow Q$ is the trace-preserving conditional expectation on qMq onto Q .

Theorem: If (M, τ) has the local approximation property, then M is hyperfinite.

Proof. Let $\epsilon > 0$ and $F \subseteq B_M$ finite. Suppose that q satisfies (i) and (ii) in the case of $e = 1$.

We observe that for all $v \in F$, the vectors

$$(1-q)vq, qv(1-q), qvq - E(v), (1-q)v(1-q)$$

are mutually orthogonal. Therefore, for all $v \in F$, we have

$$\begin{aligned} \|v - (1-q)v(1-q) - E(qvq)\|_{\tau}^2 &= \|qvq - E(v)\|_{\tau}^2 + \|(1-q)vq\|_{\tau}^2 + \|qv(1-q)\|_{\tau}^2 \\ &= \|qvq - E(v)\|_{\tau}^2 + \|[v, q]q\|_{\tau}^2 + \|q[v, q]\|_{\tau}^2 \\ &< 3\epsilon^2 \|q\|_{\tau}^2. \end{aligned}$$

Consider the partially ordered set $\mathcal{F} \subseteq \{(q, Q) \mid 0 \neq q \in P(M), Q \subseteq qMq\}$ that has $q_1 q_2 = 0$ for all $(q_1, Q_1), (q_2, Q_2) \in \mathcal{F}$ and Q finite-dimensional for all $(q, Q) \in \mathcal{F}$; using the local approximation property, we have

$$\|x - (1-\tilde{q})x(1-\tilde{q}) - E_{\tilde{Q}}(qxq)\|_{\tau} < \epsilon \|\tilde{q}\|_{\tau},$$

where we set

$$\begin{aligned} \tilde{q} &= \sum_{(q, Q) \in \mathcal{F}} q \\ \tilde{Q} &= \bigoplus_{(q, Q) \in \mathcal{F}} Q. \end{aligned}$$

The family $\mathcal{F}$ is nonempty, partially ordered by inclusion, and every increasing sequence in $\mathcal{F}$ has an upper bound in $\mathcal{F}$ by the normality of the conditional expectation.

Therefore, we have a maximal family $\{(q_j, Q_j)\}_{j \in J} \subseteq \mathcal{F}$, and set

$$\begin{aligned} q' &= \sum_{j \in J} q_j \\ Q' &= \bigoplus_{j \in J} Q_j. \end{aligned}$$

We will show that $q' = 1$. Suppose not. Then, $1 - q' \neq 0$, so there is some nonzero projection $p \leq 1 - q'$ and a finite-dimensional subalgebra $Q_p \subseteq pMp$ satisfying

$$\begin{aligned} \| [p, (1 - q)x(1 - q)] \|_\tau &< \varepsilon \| p \|_\tau \\ \| pxp - E_{Q_p}(pxp) \|_\tau &< \varepsilon \| p \|_\tau. \end{aligned}$$

Then, from the earlier discussion, we have

$$\| (1 - q)x(1 - q) - (1 - (q + p))x(1 - (q + p)) - E_{Q_p}(pxp) \|_\tau < \varepsilon \sqrt{3} \| p \|_\tau.$$

Set

$$\begin{aligned} Q_0 &= Q' \oplus Q_p \\ \widetilde{F}' &= \{q_j\}_{j \in J} \cup \{p\}. \end{aligned}$$

This gives

$$\begin{aligned} \| x - (1 - (q' + p))x(1 - (q' + p)) - E_{Q_0}((q' + p)x(q' + p)) \|_\tau^2 &= \| x - (1 - (q' + p))x(1 - (q' + p)) - E_{Q_0}(q'xq') + E_{Q_p}(pxp) \|_\tau^2 \\ &= \| (1 - q')x(1 - q') - (1 - (q' + p))x(1 - (q' + p)) - E_{Q_p}(pxp) \|_\tau^2 \\ &\quad + \| x - (1 - q')x(1 - q') - E_{Q_0}(q'xq') \|_\tau^2 \\ &< C \varepsilon \| q' + p \|_\tau^2 \end{aligned}$$

for some constant C . This contradicts the maximality of our family $\{(q_j, Q_j)\}_{j \in J}$. Since Q' is hyperfinite, this gives our desired result. $\square$

References

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