

Positive Maps

We will start by focusing our discussion of positive maps on a subclass of linear subspaces of C^* -algebras.

Definition: Let $\mathcal{A}$ be a C^* -algebra, and let $\mathcal{S} \subseteq \mathcal{A}$ be a self-adjoint linear subspace that contains 1. We call such an $\mathcal{S}$ an *operator system*.

Note that if h is a self-adjoint element of $\mathcal{S}$, then it is possible to write h as the difference of two positive elements in $\mathcal{S}$,

$$h = \frac{1}{2}(\|h\|1 + h) - \frac{1}{2}(\|h\|1 - h).$$

Definition: If $\mathcal{S} \subseteq \mathcal{A}$ is an operator system, $\mathcal{B}$ is a C^* -algebra, and $\phi: \mathcal{S} \rightarrow \mathcal{B}$ is a linear map, then we say ϕ is positive if it maps positive elements of $\mathcal{S}$ to positive elements of $\mathcal{B}$.

In the special case where the C^* -algebra $\mathcal{B}$ is the complex numbers (i.e., ϕ is a positive linear functional), then we know from results in C^* -algebra theory that $\|\phi\| = \phi(1)$. If $\mathcal{B}$ is an arbitrary C^* -algebra, it turns out that ϕ is still positive, but that the bound is different.

Proposition: If $\phi: \mathcal{S} \rightarrow \mathcal{B}$ is a positive map, then $\|\phi\| \leq 2\|\phi(1)\|$.

Proof. If p is positive, then since $0 \leq p \leq \|p\|1$, it follows that $0 \leq \phi(p) \leq \|p\|\phi(1)$, so that $\|\phi(p)\| \leq \|p\|\|\phi(1)\|$.

If p_1 and p_2 are positive, then $\|p_1 - p_2\| \leq \max(\|p_1\|, \|p_2\|)$, so if h is self-adjoint in $\mathcal{S}$, we have

$$\phi(h) = \frac{1}{2}\phi(\|h\|1 + h) - \frac{1}{2}\phi(\|h\|1 - h),$$

giving

$$\begin{aligned} \|\phi(h)\| &\leq \frac{1}{2} \max(\|\phi(\|h\|1 + h)\|, \|\phi(\|h\|1 - h)\|) \\ &\leq \|h\|\|\phi(1)\|. \end{aligned}$$

Finally, if a is an arbitrary element of $\mathcal{S}$, then we may write the Cartesian decomposition $a = h + ik$, and find

$$\begin{aligned} \|\phi(a)\| &\leq \|\phi(h)\| + \|\phi(k)\| \\ &\leq 2\|a\|\|\phi(1)\|. \end{aligned}$$

□

It turns out that this bound cannot be improved upon.

Example: Consider the subspace $\mathcal{S} \subseteq C(S^1)$ spanned by $1, z, \bar{z}$. Then, we may define $\phi: \mathcal{S} \rightarrow \mathbb{M}_2$ given by

$$\phi(a + bz + c\bar{z}) = \begin{pmatrix} a & 2b \\ 2c & a \end{pmatrix}.$$

It follows that an element of $\mathcal{S}$ is positive if and only if $c = \bar{b}$ and $a \geq 2|b|$, while a self-adjoint element of $\mathbb{M}_2$ is positive if and only if its diagonal entries and determinant are positive real numbers. Therefore, it follows that ϕ is a positive map.

Yet,

$$\begin{aligned} 2\|\phi(1)\| &= 2 \\ &= \|\phi(z)\| \\ &\leq \|\phi\|, \end{aligned}$$

meaning that $\|\phi\| = 2\|\phi(1)\|$.

Lemma: Let $\mathcal{A}$ be a unital C^* -algebra, and let $p_1, \dots, p_n$ be positive elements of $\mathcal{A}$ satisfying

$$\sum_{i=1}^n p_i \leq 1.$$

If $\lambda_1, \dots, \lambda_n$ are scalars with $|\lambda_i| \leq 1$, then we have

$$\left\| \sum_{i=1}^n \lambda_i p_i \right\| \leq 1.$$

Proof. We have

$$\begin{pmatrix} \sum_{i=1}^n \lambda_i p_i & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} = \begin{pmatrix} p_1^{1/2} & \cdots & p_n^{1/2} \\ 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix} \text{diag}(\lambda_1, \dots, \lambda_n) \begin{pmatrix} p_1^{1/2} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ p_n^{1/2} & 0 & \cdots & 0 \end{pmatrix}.$$

The norm of the matrix on the left is $\left\| \sum_{i=1}^n \lambda_i p_i \right\|$, while each of the three matrices on the right have norm less than 1, since $\|a^*a\| = \|a\|^2$. $\square$

This allows us to find some further restrictions on the norms of positive maps.

Theorem: Let $\mathcal{B}$ be a unital C^* -algebra, and let X be a compact Hausdorff space with $C(X)$ the space of continuous functions on X . Let $\phi: C(X) \rightarrow \mathcal{B}$ be a positive map. Then, $\|\phi\| = \|\phi(1)\|$.

Proof. We may assume that $\phi(1) \leq 1$. Let $f \in C(X)$, $\|f\| \leq 1$, and let $\varepsilon > 0$. Choose a finite open cover $\{U_1, \dots, U_n\}$ with respective elements $x_1, \dots, x_j$ such that $|f(x) - f(x_j)| < \varepsilon$ for all $x \in U_j$, and find a partition of unity $\{p_1, \dots, p_n\}$ subordinate to this open cover. Then, setting $\lambda_j = f(x_j)$, then if $p_i(x) \neq 0$ for some i , then $x \in U_i$, and $|f(x) - \lambda_i| < \varepsilon$. Therefore, for any $x \in X$,

$$\left| f(x) - \sum_{i=1}^n \lambda_i p_i(x) \right| \leq \sum_{i=1}^n |f(x) - \lambda_i| p_i(x) < \varepsilon.$$

By the lemma, we have that $\left\| \sum_{i=1}^n \lambda_i \phi(p_i) \right\| \leq 1$, so

$$\begin{aligned} \|\phi(f)\| &\leq \left\| \phi\left(f - \sum_{i=1}^n \lambda_i p_i\right) \right\| + \left\| \sum_{i=1}^n \lambda_i \phi(p_i) \right\| \\ &< 1 + \varepsilon \|\phi\|. \end{aligned}$$

Therefore, $\|\phi\| \leq 1$. $\square$

Completely Positive Maps

Definition: If $\mathcal{B}$ is a C^* -algebra and $\phi: \mathcal{S} \rightarrow \mathcal{B}$ is a linear map, then we may define $\phi_n: \mathbb{M}_n(\mathcal{S}) \rightarrow \mathbb{M}_n(\mathcal{B})$ by $\phi_n((a_{ij})_{i,j}) = (\phi(a_{ij}))_{i,j}$.

We call ϕ n -positive if ϕ_n is positive, and we call ϕ completely positive if ϕ is n -positive for all n .

We call ϕ completely bounded if

$$\|\phi\|_{\text{cb}} := \sup_n \|\phi_n\|$$

is finite. If $\|\phi\|_{\text{cb}} \leq 1$, then we call ϕ completely contractive.

Lemma: Let $\mathcal{A}$ be a unital C^* -algebra, and let $a, b \in \mathcal{A}$. Then,

(i) $\|a\| \leq 1$ if and only if

$$\begin{pmatrix} 1 & a \\ a^* & 1 \end{pmatrix}$$

is positive in $M_2(\mathcal{A})$.

(ii) The matrix

$$\begin{pmatrix} 1 & a \\ a^* & b \end{pmatrix}$$

is positive in $M_2(\mathcal{A})$ if and only if $a^*a \leq b$.

Proof. Faithfully represent $\mathcal{A}$ on $\mathcal{H}$ via $\pi: \mathcal{A} \rightarrow B(\mathcal{H})$, and set $A = \pi(a)$. Then, if $\|A\| \leq 1$, for any $x, y \in \mathcal{H}$, we have

$$\begin{aligned} \left\langle \begin{pmatrix} 1 & A \\ A^* & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} \right\rangle &= \langle x, y \rangle + \langle Ay, x \rangle + \langle x, Ay \rangle + \langle y, y \rangle \\ &\geq \|x\|^2 - 2\|A\|\|x\|\|y\| + \|y\|^2 \\ &\geq 0. \end{aligned}$$

Conversely, if $\|A\| > 1$ then there are unit vectors x and y such that $\langle Ay, x \rangle < -1$, so the inner product above would be negative.

Now, if we let $B = \pi(b)$, then if we let $B \geq A^*A$, then $B - A^*A \geq 0$, so that $\langle By, y \rangle \geq \langle Ay, Ay \rangle$ for all $y \in \mathcal{H}$, meaning that for all $x, y \in \mathcal{H}$, we have

$$\begin{aligned} \left\langle \begin{pmatrix} 1 & A \\ A^* & B \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} \right\rangle &= \langle x, x \rangle + \langle Ax, y \rangle + \langle A^*x, y \rangle + \langle By, y \rangle \\ &\geq \langle x, x \rangle + 2\operatorname{Re}\langle Ax, y \rangle + \langle Ay, Ay \rangle \\ &\geq \|x\|^2 + \|Ay\|^2 - 2\|Ay\|\|x\| \\ &\geq 0. \end{aligned}$$

In the case that $B \not\geq A^*A$, then there is some unit vector y such that $\langle By, y \rangle < \|Ay\|^2$, which would yield the analogous outcome as in the proof of (i). $\square$

Proposition: Let $\mathcal{S}$ be an operator system, $\mathcal{B}$ a unital C^* -algebra, and $\phi: \mathcal{S} \rightarrow \mathcal{B}$ a unital 2-positive map. Then, ϕ is contractive.

Proof. Let $a \in \mathcal{S}$ be with $\|a\| \leq 1$. Then,

$$\phi_2 \left(\begin{pmatrix} 1 & a \\ a^* & 1 \end{pmatrix} \right) = \begin{pmatrix} 1 & \phi(a) \\ \phi(a)^* & 1 \end{pmatrix}$$

is positive, so $\|\phi(a)\| \leq 1$. $\square$

Proposition: Let $\mathcal{A}$ and $\mathcal{B}$ be unital C^* -algebras, $\phi: \mathcal{A} \rightarrow \mathcal{B}$ a unital 2-positive map. Then, $\phi(a)^*\phi(a) \leq \phi(a^*a)$ for all $a \in \mathcal{A}$.

Proof. We have

$$\begin{pmatrix} 1 & a \\ 0 & 0 \end{pmatrix}^* \begin{pmatrix} 1 & a \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & a \\ a^* & a^*a \end{pmatrix}$$

$$\geq 0,$$

so

$$\begin{pmatrix} 1 & \phi(a) \\ \phi(a) & \phi(a^*a) \end{pmatrix} \geq 0.$$

Using part (ii) of the above lemma gives our desired result. $\square$

Proposition: Let $\mathcal{A}$ and $\mathcal{B}$ be unital C^* -algebras, and let $\mathcal{M} \leq \mathcal{A}$ be a unital subspace, with $\mathcal{S} = \mathcal{M} + \mathcal{M}^*$. If $\phi: \mathcal{M} \rightarrow \mathcal{B}$ is unital and 2-contractive, then the map $\varphi: \mathcal{S} \rightarrow \mathcal{B}$ given by $\varphi(a + b^*) = \phi(a) + \phi(b)^*$ is contractive and 2-positive.

Proof. Since ϕ is contractive, φ is well-defined. Since $\mathbb{M}_2(\mathcal{S}) = \mathbb{M}_2(\mathcal{M}) + \mathbb{M}_2(\mathcal{M})^*$, and $(\varphi)_2 = (\varphi_2)$, then since ϕ_2 is contractive, φ_2 is positive and φ is contractive. $\square$

Proposition: Let $\mathcal{A}$ and $\mathcal{B}$ be unital C^* -algebras, $\mathcal{M}$ a unital subspace of $\mathcal{A}$, $\mathcal{S} = \mathcal{M} + \mathcal{M}^*$. If $\phi: \mathcal{M} \rightarrow \mathcal{B}$ is unital and completely contractive, then $\varphi: \mathcal{S} \rightarrow \mathcal{B}$ as defined above is completely contractive and completely positive.

Proof. Since ϕ_n is unital and contractive, φ_n is positive, and since $\varphi_{2n} = (\varphi_n)_2$, we have φ_n is contractive. $\square$

Proposition: Let $\mathcal{S} \subseteq \mathcal{A}$ be an operator system, and let $\mathcal{B}$ be a C^* -algebra, with $\phi: \mathcal{S} \rightarrow \mathcal{B}$ completely positive. Then, ϕ is completely bounded, satisfying $\|\phi(1)\| = \|\phi\| = \|\phi\|_{\text{cb}}$.

Proof. We have $\|\phi(1)\| \leq \|\phi\| \leq \|\phi\|_{\text{cb}}$, so it suffices to show that $\|\phi\|_{\text{cb}} \leq \|\phi(1)\|$. If $A = (a_{ij})_{i,j}$ is in $\mathbb{M}_n(\mathcal{S})$ with $\|A\| \leq 1$, and I_n is the unit matrix of $\mathbb{M}_n(\mathcal{A})$, then the block matrix

$$P = \begin{pmatrix} I_n & A \\ A^* & I_n \end{pmatrix}$$

is positive, so

$$\phi_{2n}(p) = \begin{pmatrix} \phi_n(I_n) & \phi_n(A) \\ \phi_n(A)^* & \phi_n(I_n) \end{pmatrix}$$

is positive. Therefore, $\|\phi_n(A)\| \leq \|\phi_n(I_n)\| = \|\phi(1)\|$. $\square$

Dilations and Extensions

We discuss a characterization of completely positive maps, as well as a way to understand extensions of completely positive maps.

Stinespring's Dilation Theorem

Theorem (Stinespring's Dilation Theorem): Let $\mathcal{A}$ be a unital C^* -algebra, and let $\phi: \mathcal{A} \rightarrow B(\mathcal{H})$ be a completely positive map. Then, there exists a Hilbert space $\mathcal{K}$, a unital $*$ -homomorphism $\pi: \mathcal{A} \rightarrow B(\mathcal{K})$, and a bounded operator $V: \mathcal{H} \rightarrow \mathcal{K}$ with $\|\phi(1)\| = \|V\|^2$ such that

$$\phi(a) = V^* \pi(a) V.$$

Proof. Let $A \odot \mathcal{H}$ be the algebraic tensor product, and define a symmetric bilinear map

$$(a \otimes x, b \otimes y) = \langle \phi(b^*a)x, y \rangle,$$

and extend linearly, where $\langle \cdot, \cdot \rangle$ is the inner product on $\mathcal{H}$. Then, since ϕ is completely positive, it follows

that $(\cdot, \cdot)$ is positive semidefinite, with

$$\left(\sum_{j=1}^n a_j \otimes x_j, \sum_{i=1}^n a_i \otimes x_i \right) = \left\langle \phi_n \left((a_i^* a_j)_{i,j} \right) \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \right\rangle \geq 0.$$

Since positive semidefinite bilinear forms satisfy the Cauchy–Schwarz inequality, we may define the subspace

$$\begin{aligned} \mathcal{N} &= \{u \in \mathcal{A} \otimes \mathcal{H} \mid (u, u) = 0\} \\ &= \{u \in \mathcal{A} \otimes \mathcal{H} \mid (u, v) = 0 \text{ for all } v \in \mathcal{A} \otimes \mathcal{H}\}, \end{aligned}$$

with an induced bilinear form on the quotient space $\mathcal{A} \otimes \mathcal{H} / \mathcal{N}$ defined by

$$\langle u + \mathcal{N}, v + \mathcal{N} \rangle = (u, v).$$

Define $\mathcal{K}$ to be the Hilbert space completion of $\mathcal{A} \otimes \mathcal{H} / \mathcal{N}$. Now, define a linear map $\pi(a): \mathcal{A} \otimes \mathcal{H} \rightarrow \mathcal{A} \otimes \mathcal{H}$ by

$$\pi(a) \left(\sum_{i=1}^n a_i \otimes x_i \right) = \sum_{i=1}^n (aa_i) \otimes x_i.$$

We have that the inequality in $\mathbb{M}_n(\mathcal{A})$ given by

$$(a_i^* a^* a a_j)_{i,j} \leq \|a^* a\| (a_i^* a_j)_{i,j}$$

is satisfied, giving

$$\begin{aligned} \left(\pi(a) \left(\sum_{j=1}^n a_j \otimes x_j \right), \pi(a) \left(\sum_{i=1}^n a_i \otimes x_i \right) \right) &= \sum_{i,j=1}^n \langle \phi(a_i^* a^* a a_j) x_j, x_i \rangle \\ &\leq \|a^* a\| \sum_{i,j=1}^n \langle \phi(a_i^* a_j) x_j, x_i \rangle \\ &= \|a\|^2 \left(\sum_{j=1}^n a_j \otimes x_j, \sum_{i=1}^n a_i \otimes x_i \right). \end{aligned}$$

Therefore, $\pi(a)$ is invariant under $\mathcal{N}$, so induces a quotient map on $\mathcal{A} \otimes \mathcal{H} / \mathcal{N}$, which we will also denote by $\pi(a)$. We have that (this new) $\pi(a)$ is bounded with $\|\pi(a)\| \leq \|a\|$, meaning that it extends to a bounded linear map on $\mathcal{K}$.

We define $V: \mathcal{H} \rightarrow \mathcal{K}$ by

$$V(x) = 1 \otimes x + \mathcal{N}.$$

Then,

$$\begin{aligned} \|Vx\|^2 &= (1 \otimes x, 1 \otimes x) \\ &= \langle \phi(1)x, x \rangle \\ &\leq \|\phi(1)\| \|x\|^2, \end{aligned}$$

and $\|V\|^2 = \|\phi(1)\|$.

Finally, we observe that

$$\begin{aligned} \langle V^* \pi(a) Vx, y \rangle &= (\pi(a) 1 \otimes x, 1 \otimes y) \\ &= \langle \phi(a)x, y \rangle \end{aligned}$$

for all x and y , so $V^* \pi(a) V = \phi(a)$. □

We observe that if ϕ is unital, then V is an isometry, and we may identify $\mathcal{H}$ with the subspace $V\mathcal{H}$ of $\mathcal{K}$, and that $\phi(a) = P\pi(a)P$, where P is the projection onto $\mathcal{H}$. In particular, this means that every unital completely positive map is the compression of a $*$ -homomorphism.

This construction is very similar to the GNS representation, and we call the triple $(\pi, V, \mathcal{K})$ a Stinespring representation for ϕ .

If $\mathcal{K}_1 = [\pi(\mathcal{A})V\mathcal{H}]$ denotes the closed linear span of the image of $\pi(\mathcal{A})V$, then the restriction of π to $\mathcal{K}_1$ defines a $*$ -homomorphism. If there is a Stinespring representation $(\pi_1, V, \mathcal{K}_1)$ that satisfies $\mathcal{K}_1 = [\pi_1(\mathcal{A})V\mathcal{H}]$, then we call such a triple a *minimal* Stinespring representation.

The minimal Stinespring representation is unique up to unitary equivalence.

Proposition: Let $(\pi, V, \mathcal{H})$ be a minimal Stinespring dilation of a completely positive contraction $\phi: A \rightarrow B(\mathcal{H})$. Then, there exists a $*$ -homomorphism $\rho: \phi(A)' \rightarrow \pi(A)'$ such that $\phi(a)x = V^*\pi(a)\rho(x)V$ for all $a \in A$ and all $x \in \phi(A)'$.

One of the most important applications of the Stinespring theorem is showing a version of the Cauchy–Schwarz inequality, as well as a bimodularity property.

Proposition: Let A and B be C^* -algebras, with $\phi: A \rightarrow B$ a completely positive contraction.

- (i) The inequality $\phi(a)^*\phi(a) \leq \phi(a^*a)$ holds for all $a \in A$.
- (ii) If, for some $a \in A$, we have $\phi(a^*a) = \phi(a)^*\phi(a)$ and $\phi(aa^*) = \phi(a)\phi(a)^*$, then we have $\phi(ba) = \phi(b)\phi(a)$ and $\phi(ab) = \phi(a)\phi(b)$ for all $b \in A$.
- (iii) The subspace

$$A_\phi = \{a \in A \mid \phi(a^*a) = \phi(a)^*\phi(a) \text{ and } \phi(aa^*) = \phi(a)\phi(a)^*\}$$

is a C^* -subalgebra of A .

Proof. Let $B \subseteq B(\mathcal{H})$ be a faithful $*$ -representation, and $(\pi, V, \mathcal{K})$ a Stinespring dilation of $\phi: A \rightarrow B \subseteq B(\mathcal{H})$. For every $a \in A$, we have

$$\begin{aligned} \phi(a^*a) - \phi(a)^*\phi(a) &= V^*\pi(a)^*(1 - VV^*)\pi(a)V \\ &\geq 0 \end{aligned}$$

as V is a contraction. Therefore, (i) follows.

Moreover, equality holds if and only if $(1 - VV^*)^{1/2}\pi(a)V = 0$, meaning that

$$\begin{aligned} \phi(ba) - \phi(b)\phi(a) &= V^*\pi(b)(1 - VV^*)\pi(a)V \\ &= 0 \end{aligned}$$

for every $b \in A$. Symmetry gives (ii), and (iii) follows from (ii). □

The C^* -subalgebra A_ϕ is called the *multiplicative domain* of ϕ .

Arveson's Extension Theorem

We now begin by understanding relations between completely positive maps and the matrix algebras $\mathbb{M}_n(\mathbb{C})$.

Proposition: Let A be a C^* -algebra, and let $(e_{ij})_{i,j}$ denote the matrix units of $\mathbb{M}_n(\mathbb{C})$. A map $\phi: \mathbb{M}_n(\mathbb{C}) \rightarrow A$ is completely positive if and only if the matrix $(\phi(e_{ij}))_{i,j}$ is positive in $\mathbb{M}_n(A)$.

Proof. Since $((e_{ij})_{i,j})_{i,j} \in \mathbb{M}_n(\mathbb{M}_n(\mathbb{C}))$ (the matrix of matrix units) is positive, as it is a multiple of a rank one projection, the forward direction follows immediately.

In the reverse direction, let $a = (a_{ij})_{i,j} = (\varphi(e_{ij}))_{i,j}$ in $\mathbb{M}_n(A)$ be positive, and let $(b_{ij})_{i,j} = a^{1/2}$ be the square root. By definition, then have

$$\varphi(e_{ij}) = \sum_{k=1}^n b_{ki}^* b_{kj}.$$

If $A \subseteq B(\mathcal{H})$ is a faithful representation, define $V : \mathcal{H} \rightarrow \ell_2^n \otimes \ell_2^n \otimes \mathcal{H}$ by

$$V\xi = \sum_{j,k=1}^n \zeta_j \otimes \zeta_k \otimes b_{kj}\xi,$$

where $\{\zeta_j\}_{j=1}^n$ is the standard basis for ℓ_2^n . Then, if $T = (t_{ij})_{i,j} \in \mathbb{M}_n(\mathbb{C})$, we have

$$\begin{aligned} \langle V^*(T \otimes 1 \otimes 1)V\eta, \xi \rangle &= \langle (T \otimes 1 \otimes 1)V\eta, V\xi \rangle \\ &= \sum_{i,j,k,\ell=1}^n \langle T\zeta_j, \zeta_i \rangle \langle \zeta_k, \zeta_\ell \rangle \langle b_{kj}\eta, b_{\ell i}\xi \rangle \\ &= \left\langle \varphi\left(\sum_{i,j=1}^n t_{ij}e_{ij}\right)\eta, \xi \right\rangle \\ &= \langle \varphi(T)\xi, \eta \rangle, \end{aligned}$$

for every $\xi, \eta \in \mathcal{H}$. Thus, $\varphi(T) = V^*(T \otimes 1 \otimes 1)V$ for every $T \in \mathbb{M}_n(\mathbb{C})$, so φ is completely positive. $\square$

In particular, for fixed $a_1, \dots, a_n \in A$, we may define $\varphi : \mathbb{M}_n(\mathbb{C}) \rightarrow A$ by $\varphi(e_{ij}) = a_i a_j^*$, and observe that this gives a completely positive map, as

$$\begin{aligned} (\varphi(e_{ij})) &= \begin{pmatrix} a_1 a_1^* & a_1 a_2^* & \cdots & a_1 a_n^* \\ a_2 a_1^* & a_2 a_2^* & \cdots & a_2 a_n^* \\ \vdots & \vdots & \ddots & \vdots \\ a_n a_1^* & a_n a_2^* & \cdots & a_n a_n^* \end{pmatrix} \\ &= \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ a_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_n & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ a_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_n & 0 & \cdots & 0 \end{pmatrix}^* \end{aligned}$$

There is a similar reverse characterization. Given a linear map $\varphi : A \rightarrow \mathbb{M}_n(\mathbb{C})$, define a linear functional $\hat{\varphi} : \mathbb{M}_n(A) \rightarrow \mathbb{C}$ by

$$\hat{\varphi}((a_{ij})_{i,j}) = \sum_{i,j=1}^n (\varphi(a_{ij}))_{i,j}$$

Proposition: A map $\varphi : A \rightarrow \mathbb{M}_n(\mathbb{C})$ is completely positive if and only if $\hat{\varphi}$ is positive on $\mathbb{M}_n(A)$.

Proof. Let $\{\zeta_i\}_{i=1}^n$ be the standard orthonormal basis for ℓ_2^n , and let

$$\begin{aligned} \zeta &= \begin{pmatrix} \zeta_1 \\ \vdots \\ \zeta_n \end{pmatrix} \\ &\in (\ell_2^n)^n. \end{aligned}$$

We have

$$\hat{\phi}\left(\left(a_{ij}\right)_{i,j}\right)=\left\langle\varphi_n\left(\left(a_{ij}\right)_{i,j}\right)\zeta,\zeta\right\rangle$$

whenever $\left(a_{ij}\right)_{i,j} \in \mathbb{M}_n(A)$, so if φ_n is positive, then so too is $\hat{\phi}$.

In the reverse direction, let $\hat{\phi}$ be positive, and let $L_2(\mathbb{M}_n(A), \hat{\phi})$ be the GNS representation with cyclic vector ξ . If $\left(e_{ij}\right)_{i,j}$ denotes the system of matrix units for $\mathbb{M}_n(\mathbb{C})$, we may view them as elements in $\mathbb{M}_n(A)$ by tensoring with 1_A . Define the operator $V: \ell_2^n \rightarrow L_2(\mathbb{M}_n(A), \hat{\phi})$ by

$$V\zeta_j = \pi(e_{1j})\xi.$$

Then, we have

$$\varphi(a) = V^*\pi(\text{diag}(a, \dots, a))V,$$

so that φ is completely positive. $\square$

Lemma: Let $E \subseteq A$ be an operator system, and let $\psi: E \rightarrow \mathbb{C}$ be a positive linear functional. Then, $\|\psi\| = \psi(1)$, and any norm-preserving extension of ψ to A is positive.

Proof. Let $x \in E$ be such that $\|x\| \leq 1$ and $|\psi(x)|$ is close to $\|\psi\|$. We may assume that $0 < \psi(x) \in \mathbb{R}$ by multiplying by the sign of $\psi(x)$. Since positive maps are self-adjoint, we have

$$\psi(x) = \frac{1}{2}\psi(x + x^*),$$

so we may assume x is self-adjoint. Yet, the operator inequality $x \leq \|x\|1$ induces

$$\psi(x) \leq \psi(1)\|x\|.$$

Since any functional satisfying $\psi(1) = \|\psi\|$ is positive, we have that any extension to A is positive. $\square$

Corollary: If $E \subseteq A$ is an operator system, and $\varphi: E \rightarrow \mathbb{M}_n(\mathbb{C})$ is a completely positive map, then φ extends to a completely positive map $A \rightarrow \mathbb{M}_n(\mathbb{C})$.

Proof. We may define the positive linear functional $\hat{\phi}$ on $\mathbb{M}_n(E)$, which extends to a positive linear functional on $\mathbb{M}_n(A)$. Applying the correspondence in reverse gives our desired extension. $\square$

Theorem: Let A be a unital C^* -algebra, and let $E \subseteq A$ be an operator system. Then, every contractive completely positive map $\varphi: E \rightarrow B(\mathcal{H})$ extends to a contractive completely positive map $\bar{\varphi}: A \rightarrow B(\mathcal{H})$.

Proof. Let $(P_i)_{i \in I}$ be an increasing net of finite-rank projections that converge to identity in SOT. For each i , consider the map $\varphi_i: E \rightarrow P_i B(\mathcal{H}) P_i$ given by $\varphi_i(e) = P_i \varphi(e) P_i$, which can be viewed as taking values in a matrix algebra.

We may extend φ_i to be defined on all of A , and letting φ_i take values in $B(\mathcal{H})$, we use point-ultraweak compactness of the unit ball in $B(A, B(\mathcal{H}))$ to find a cluster point $\Phi: A \rightarrow B(\mathcal{H})$. This map Φ is completely positive and extends φ . $\square$

Nuclearity and Exactness

Definition: We call a map $\theta: A \rightarrow B$ between C^* -algebras *nuclear* if there are contractive completely positive maps $\varphi_n: A \rightarrow \mathbb{M}_{k(n)}(\mathbb{C})$ and $\psi_n: \mathbb{M}_{k(n)}(\mathbb{C}) \rightarrow B$ such that $\psi_n \circ \varphi_n \rightarrow \theta$ pointwise in the norm topology:

$$\|\psi_n \circ \varphi_n(a) - \theta(a)\| \rightarrow 0$$

for all $a \in A$.

Definition: If A is a C^* -algebra, and N is a von Neumann algebra, then a map $\theta: A \rightarrow N$ is called *weakly nuclear* if there exist contractive completely positive maps $\varphi_n: A \rightarrow \mathbb{M}_{k(n)}(\mathbb{C})$ and $\psi_n: \mathbb{M}_{k(n)}(\mathbb{C}) \rightarrow N$ such that $\psi_n \circ \varphi_n \rightarrow \theta$ pointwise in the ultraweak topology:

$$\eta(\psi_n \circ \varphi_n(a)) \rightarrow \eta(\theta(a))$$

for all $a \in A$ and normal functionals $\eta \in N_*$.

By uniqueness of preduals, if $N \subseteq B(\mathcal{H})$ is a faithful normal representation, then it suffices to observe that $\psi_n \circ \varphi_n \rightarrow \theta$ pointwise in the ultraweak topology if and only if

$$\langle \psi_n \circ \varphi_n(a)v, w \rangle \rightarrow \langle \theta(a)v, w \rangle$$

for all $a \in A$ and $v, w \in \Omega$ for some collection Ω of vectors whose linear span is dense in $\mathcal{H}$.

One of the interesting aspects of nuclear maps is that whether a map is nuclear or not depends on the range.

Proposition: Let $M \subseteq B(H)$ be a von Neumann algebra. The natural inclusion map $M \hookrightarrow B(H)$ is always weakly nuclear.

Proof. Let $\{P_i\}_{i \in I}$ be a net of finite rank projections increasing to the identity. If P_i has rank $k(i)$, then we may define maps $\varphi_i: M \rightarrow \mathbb{M}_{k(i)}(\mathbb{C}) \cong P_i B(H) P_i$ by compression, and let $\psi_i: \mathbb{M}_{k(i)}(\mathbb{C}) \rightarrow B(H)$ be natural inclusion maps.

Since the predual of $B(H)$ is the trace class operators, we have that these maps converge weakly to the identity on $B(H)$, it follows that $M \hookrightarrow B(H)$ is weakly nuclear. $\square$

Proposition: A map $\theta: A \rightarrow B$ is nuclear if and only if for every finite $F \subseteq A$ and every $\varepsilon > 0$, there exist $n \in \mathbb{N}$ and contractive completely positive maps $\varphi: A \rightarrow \mathbb{M}_n(\mathbb{C})$, $\psi: \mathbb{M}_n(\mathbb{C}) \rightarrow B$ such that $\|\theta(a) - \psi \circ \varphi(a)\| < \varepsilon$ for all $a \in F$.

Proof. Define the set $\mathcal{F} = \{(F, \varepsilon) \mid F \subseteq A \text{ finite}, \varepsilon > 0\}$, directed by

$$(F_1, \varepsilon_1) \leq (F_2, \varepsilon_2) \Leftrightarrow F_1 \subseteq F_2 \text{ and } \varepsilon_2 \leq \varepsilon_1.$$

It can then be verified that convergence in the point-norm topology in the definition for nuclearity is equivalent to convergence via this directed set, and vice versa. $\square$

Note that the point-norm topology is (effectively) equivalent to the strong operator topology, while the point-weak topology is (effectively) equivalent to the weak operator topology.

Definition: A C^* -algebra A is called *nuclear* if the identity map is nuclear.

Definition: A C^* -algebra A is called *exact* if there exists a faithful representation $\pi: A \rightarrow B(H)$ such that π is nuclear.

Definition: A von Neumann algebra M is called *semidiscrete* if the identity map is weakly nuclear.

We will show now that if the double dual of a C^* -algebra is semidiscrete, then the C^* -algebra is nuclear.

Lemma: Let A be a Banach space, and let $B(A)$ be the space of all bounded linear maps from A to A , and let $C \subseteq B(A)$ be a convex set. Then, the point-weak and point-norm closures of C coincide.

Proof. Let $T \in B(A)$ be a map such that $(T_i)_i \subseteq C$ converges in the point-weak topology — i.e., for all $a \in A$ and all $\eta \in A^*$,

$$\eta(T_i a) \rightarrow \eta(T a).$$

We must show that for any finite subset $F = \{a_1, \dots, a_k\}$ of A and every $\varepsilon > 0$, there is $S \in C$ such that $\|S a_j - T a_j\| < \varepsilon$ for all $1 \leq j \leq k$.

Since the net of maps $(T_i \oplus \dots \oplus T_i)_i$ converges in the point-weak topology to $T \oplus \dots \oplus T$, we have that the element $T a_1 \oplus \dots \oplus T a_k$ belongs to the weak closure of $(T_i a_1 \oplus \dots \oplus T_i a_k)$, so by the Hahn–Banach separation, it belongs to the norm closure of the convex hull. Therefore, we can find a convex combination of the T_i that is close in norm to T on all of F . $\square$

Definition: We say a completely positive map $\theta: A \rightarrow A$ is *factorable* if there exist completely positive maps $\varphi: A \rightarrow \mathbb{M}_n(\mathbb{C})$ and $\psi: \mathbb{M}_n(\mathbb{C}) \rightarrow A$ such that $\theta = \psi \circ \varphi$.

Proposition: The set of factorable maps is convex.

Proof. Let $\theta_1, \theta_2: A \rightarrow A$ be factorable maps with factorizations given by $(\varphi_i, \psi_i, \mathbb{M}_{n_i}(\mathbb{C}))$. We observe then that $\lambda\theta_1 + (1 - \lambda)\theta_2$ factorizes through $\mathbb{M}_{n_1}(\mathbb{C}) \oplus \mathbb{M}_{n_2}(\mathbb{C})$, given by $\varphi_1 \oplus \varphi_2$ and $\lambda\psi_1 + (1 - \lambda)\psi_2$ respectively.

The set of positive operators is a cone, so this map is completely positive, and we may embed $\mathbb{M}_{n_1}(\mathbb{C}) \oplus \mathbb{M}_{n_2}(\mathbb{C}) \hookrightarrow \mathbb{M}_{n_3}(\mathbb{C})$ to find this desired factorization. $\square$

Note that this also shows that the space of contractive factorable maps is also convex.

Proposition: If A^{**} is semidiscrete, then A is nuclear.

Proof. Suppose A is unital, fix finite subsets $F \subseteq A$ and $Z \subseteq (A^{**})_* = A^*$, and $\varepsilon > 0$. By some previous lemma, it suffices to show that id_A belongs to the point-weak closure of the contractive factorable maps. That is, there are contractive completely positive maps $\varphi: A \rightarrow \mathbb{M}_n(\mathbb{C})$ and $\psi: \mathbb{M}_n(\mathbb{C}) \rightarrow A$ such that, for all $a \in F$ and $\eta \in Z$, we have

$$|\eta(\psi \circ \varphi(a)) - \eta(a)| < \varepsilon.$$

Since A^{**} is semidiscrete, we may find unital completely positive maps $\varphi: A^{**} \rightarrow \mathbb{M}_n(\mathbb{C})$ and $\psi': \mathbb{M}_n(\mathbb{C}) \rightarrow A^{**}$ such that $|\eta(\psi' \circ \varphi(a)) - \eta(a)| < \varepsilon$ for all $a \in F$ and $\eta \in Z$. $\square$

Applications

There are a variety of ways to use understanding of completely positive maps to understand ideas in von Neumann algebra theory.

Conditional Expectations

This subsection is copied from [projections](#) notes.

Definition: If A is a unital C^* -algebra, and $B \subseteq A$ is a unital C^* -subalgebra, then a conditional expectation from A to B is a unital completely positive map $E: A \rightarrow B$ with $E|_B = \text{id}$.

From [Pau02, Corollary 3.19], every conditional expectation is bimodular — that is, $E(axb) = aE(x)b$ for all $a, b \in B$ and $x \in A$.

It turns out that the stated properties are defining for conditional expectations.

Theorem: Let A be a unital C^* -algebra, $B \subseteq A$ a unital C^* -subalgebra, and $E: A \rightarrow B$ with $E|_B = \text{id}$ and $\|E\| \leq 1$. Then, E is a conditional expectation.

Proof. Start by considering the case that A is a von Neumann algebra and B is a von Neumann subalgebra.

Then, for any $p \in P(B)$ and $x \in A$, we have $(1 - p)E(px) = E((1 - p)E(px))$, so for all $t > 0$,

$$\begin{aligned} (1 + t)^2 \|pE((1 - p)x)\|^2 &= \|pE((1 - p)x + tpE((1 - p)x))\|^2 \\ &\leq \|(1 - p)x + tpE((1 - p)x)\|^2 \\ &\leq \|(1 - p)x\|^2 + t^2 \|pE((1 - p)x)\|^2, \end{aligned}$$

so for all $t > 0$, we have $(1 + 2t)\|pE((1 - p)x)\|^2 \leq \|(1 - p)x\|^2$, so that $pE((1 - p)x) = 0$. Since both p and $1 - p$ are in B , we have that $pE(x) = pE(px) = E(px)$. Since the span of the projections is norm-dense in B , we have $E(yx) = yE(x)$ for all $y \in B$ and all $x \in A$. Therefore, we have that E is B -bimodular.

Since E is a unital contraction, we have that E is positive. If $(a_{ij})_{i,j} \in \mathbb{M}_n(A)_+$ and $x_1, \dots, x_n \in B$, then we

have

$$\sum_{i,j=1}^n x_i^* E(a_{ij}) x_j = E \left(\sum_{i,j=1}^n x_i^* a_{ij} x_j \right) \geq 0,$$

so $E^{(n)} \left((a_{ij})_{i,j} \right) \geq 0$, meaning E is completely positive.

In the general case, we embed A into its double dual, A^{**} , which is a von Neumann algebra, and similarly for B , and use the fact that $E^{**}: A^{**} \rightarrow B^{**}$ is equal to E on the subalgebras A and B . $\square$

Theorem: Let M be a finite von Neumann algebra with trace τ , and let $N \subseteq M$ be a von Neumann subalgebra. Then, there exists a unique normal conditional expectation $E: M \rightarrow N$ with $\tau \circ E = \tau$.

Proof. Let $e_N \in B(L_2(M, \tau))$ be the projection onto $L_2(N, \tau) \subseteq L_2(M, \tau)$. Let J be the conjugation operator on $L_2(M, \tau)$, which we can also view as the conjugation operator on $L_2(N, \tau)$.

We have $N' \cap B(L_2(N, \tau))$ from a basic result on the standard representation (see [orbit equivalence notes](#)). Since $L_2(N, \tau)$ is invariant under the operation JNJ , we must have $e_N(Jy) = (Jy)e_N$ for all $y \in N$.

For any $x \in M$ and $y \in N$, we get

$$\begin{aligned} e_N x e_N J y J &= e_N x J y J \\ &= e_N x J y J x e_N \\ &= J y J e_N x e_N, \end{aligned}$$

so $e_N x e_N \in (JNJ)' = N$, and we may denote this operator by $E(x)$. We have that $E: M \rightarrow N$ is normal, unital, completely positive, and $E|_N = \text{id}$, so E is a normal conditional expectation. Furthermore, for any $x \in M$, we have

$$\begin{aligned} \tau(E(x)) &= \langle e_N x e_N 1_\tau, 1_\tau \rangle \\ &= \langle x 1_\tau, 1_\tau \rangle \\ &= \tau(x), \end{aligned}$$

where 1_τ denotes the cyclic vector on the GNS representation.

If $\hat{E}$ were another trace-preserving conditional expectation, then for any $x \in M$ and $y \in N$, we would have

$$\begin{aligned} \tau(\hat{E}(x)y) &= \tau(\hat{E}(xy)) \\ &= \tau(xy) \\ &= \tau(E(xy)) \\ &= \tau(E(x)y), \end{aligned}$$

meaning that $\hat{E} = E$ as x is arbitrary and τ is faithful. $\square$

Amenability

Now we can discuss how completely positive maps relate to amenability.

Recall that a (discrete, countable) group G is called *amenable* if there is a state $m: \ell_\infty(G) \rightarrow \mathbb{C}$ satisfying $m(\lambda_s f) = m(f)$ for all $s \in G$, where $\lambda_s f(t) = f(s^{-1}t)$. There are a variety of equivalent characterizations, which are discussed in depth in [this document](#).

Proposition: Let G be a group. The following are equivalent:

- (i) G is amenable;

(ii) there exists a net $(\xi_i)_{i \in I}$ of unit vectors in $\ell_2(G)$ such that, for all $g \in G$, we have

$$\lim_{i \in I} \|\lambda_g \xi_i - \xi_i\|_{\ell_2} = 0;$$

(iii) there exists a net of finitely supported positive definite functions on G that converge pointwise to 1;

(iv) there exists a net $(E_i)_{i \in I}$ of nonempty finite subsets of G such that

$$\lim_{i \in I} \frac{|g E_i \Delta E_i|}{|E_i|} = 0.$$

Our goal will be to use the theory of completely positive maps to introduce other characterizations of amenability both for groups and von Neumann algebras.

Proposition: Let G be a group, and let $M = L(G)$ be its group von Neumann algebra. Then, G is amenable if and only if there exists a conditional expectation $E: B(L_2(M)) \rightarrow M$.

Proof. Suppose E exists. If $f \in \ell_\infty(G)$, let M_f be the operator of multiplication by f on $\ell_2(G)$. Set $m(f) = \tau(E(M_f))$, where $\tau = \langle \cdot, \delta_e, \delta_e \rangle$ is the canonical trace on M . Since $u_s M_f u_s^* = M_{\lambda_s f}$ for every $s \in G$, we have that m is left invariant.

Now, assume G is amenable, and let m be a mean on $\ell_\infty(G)$. Given $\xi, \eta \in \ell_2(G)$, and $T \in B(\ell_2(M))$, define the function

$$f_{\xi, \eta}^T(s) = \langle \xi, \rho(s) T \rho(s^{-1}) \eta \rangle,$$

where ρ is the right regular representation of G . We have that $f_{\xi, \eta}^T$ is a bounded function on G satisfying

$$|f_{\xi, \eta}^T(s)| \leq \|T\| \|\xi\| \|\eta\|.$$

This allows us to define a continuous sesquilinear form

$$(\xi, \eta) = m(f_{\xi, \eta}^T)$$

on $\ell_2(G)$. By the structure of sesquilinear forms, we have a unique operator $E(T)$ satisfying

$$\langle \xi, E(T) \eta \rangle = m(f_{\xi, \eta}^T).$$

Since m is invariant under the left-regular action, it is invariant under the right-regular action, so

$$\rho(s) E(T) \rho(s^{-1}) = E(T),$$

so $E(T)$ commutes with $\rho(G)$, meaning $E(T) \in L(G)$, and $E(T)$ is a conditional expectation as it is equal to identity on $L(G)$ with norm less than 1. $\square$

This property motivates the following definition.

Definition: We say a von Neumann algebra is *amenable* (or *injective*) if M can be faithfully represented as a von Neumann subalgebra of some $B(H)$ such that there exists a conditional expectation $E: B(H) \rightarrow M$.

Proposition: Let M be a von Neumann algebra. The following are equivalent:

- (i) M is injective;
- (ii) for every inclusion $A \subseteq B$ of unital C^* -subalgebras, any unital completely positive map $\phi: A \rightarrow M$ extends to a completely positive map from B to M ;
- (iii) for any $B(H)$ that contains M as a von Neumann subalgebra, there is a conditional expectation from $B(H)$ onto M .

Proof. To show (i) implies (ii), assume M is a von Neumann subalgebra of $B(H)$, and let $E: B(H) \rightarrow M$ be a conditional expectation. Extend ϕ to a completely positive map $\tilde{\phi}: B \rightarrow B(H)$ by using Arveson's extension theorem. Then, $E \circ \tilde{\phi}: B \rightarrow M$ is a completely positive extension of ϕ .

For (ii) implies (iii), if $M \subseteq B(H)$, then apply (ii) with $A = M$, $B = B(H)$, and the identity map id_M . Then, there exists a completely positive map $\phi: B(H) \rightarrow M$ whose restriction to M is id_M , which is necessarily a conditional expectation. $\square$

Example: The hyperfinite II_1 factor, R , is amenable. Recall that the hyperfinite II_1 factor is defined to be the SOT-closure of the inductive limit of $Q_n = \mathbb{M}_{2^n}(\mathbb{C})$, where we equip $Q_n \rightarrow Q_{n+1}$ with the block-diagonal embedding,

$$A \mapsto \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}.$$

Let J be the conjugation operator on $L_2(R)$. For any $n \geq 1$ and $T \in B(L_2(R))$, set

$$E_n(T) = J \left(\int_{U(2^n)} u J T J u^* d\mu(u) \right) J,$$

where μ denotes the Haar probability measure on the compact group $U(2^n)$, the group of unitary $2^n \times 2^n$ matrices. Then, $(E_n(T))_n$ is a norm-bounded sequence of operators in $B(L_2(R))$.

Take the limit of this sequence along a free ultrafilter, which allows for the existence of a unique bounded operator denoted $E(T)$ satisfying

$$\langle \xi, E(T)\eta \rangle = \lim_{\omega} \langle \xi, E_n(T)\eta \rangle$$

for all $\xi, \eta \in L_2(R)$. Observe that for any $n \geq n_0$, we have that $E_n(T) \in J(Q_{n_0})'J$, so that

$$\begin{aligned} E(T) &\in \bigcap_{n_0 \geq 1} J(Q_{n_0})'J \\ &= R, \end{aligned}$$

and $E(T) = T$ if $T \in R$. This gives that E is our desired conditional expectation.

Definition: Let (M, τ) be a tracial von Neumann algebra. A state ψ on $B(L_2(M, \tau))$ is called a *hypertrace* if $\psi(xT) = \psi(Tx)$ for every $x \in M$ and every $T \in B(L_2(M, \tau))$, with $\psi|_M = \tau$.

Proposition: Let (M, τ) be a tracial von Neumann algebra. Then, M is amenable if and only if it has a hypertrace.

Proof. If M is amenable, then since $M \subseteq B(L_2(M, \tau))$, we have a conditional expectation $E: B(L_2(M, \tau)) \rightarrow M$, so that $\tau \circ E$ is a hypertrace.

Conversely, if ψ is a hypertrace, then given $T \in B(L_2(M, \tau))_+$, we may define a positive linear functional φ on M by taking $\varphi(x) = \psi(Tx)$. For any $x \in M_+$, we then have

$$\begin{aligned} |\varphi(x)|^2 &= \left| \psi(x^{1/2} T x^{1/2}) \right|^2 \\ &\leq \psi(x^{1/2} T^2 x^{1/2}) \psi(x) \\ &\leq \|T\|^2 \psi(x)^2. \end{aligned}$$

Since $\psi|_M = \tau$, we have $\varphi(x) \leq \|T\| \tau(x)$ for any $x \in M_+$, and in particular, φ is normal. By the noncommutative Radon–Nikodym theorem, there is some $E(T) \in M_+$ such that for all $x \in M$,

$$\psi(Tx) = \tau(E(T)x).$$

We may extend E to yield a conditional expectation from $B(L_2(M, \tau)) \rightarrow M$. $\square$

Bimodules and Completely Positive Maps

There is a significant connection between completely positive maps and the notion of bimodules over von Neumann algebras.

Definition: Let M and N be finite von Neumann algebras. A Hilbert space H is said to be a M - N -bimodule if it is equipped with two commuting normal $*$ -representations $\pi: M \rightarrow B(H)$ and $\rho: N^{\text{op}} \rightarrow B(H)$. We will write

$$x \cdot \xi \cdot y = \pi(x)\rho(y^{\text{op}})\xi$$

for all $\xi \in H$, $x \in M$, and $y \in N$.

Example: The following are important examples of bimodules.

- (i) The standard representation $L_2(M)$ is an M - M -bimodule with representations of M and M^{op} given by $x \cdot \xi \cdot y = x\xi y$.
- (ii) The coarse, denoted $L_2(M) \otimes L_2(N)$, is an M - N -bimodule given by $x \cdot (\xi \otimes \eta) \cdot y = (x\xi) \otimes (\eta y)$. We have that

$$L_2(M) \otimes L_2(N) \cong \text{HS}(L_2(M), L_2(N)),$$

where $\text{HS}(L_2(M), L_2(N))$ is the bimodule of Hilbert–Schmidt operators from $L_2(M)$ to $L_2(N)$.

- (iii) If $\tau \in \text{aut}(M)$ is a trace-preserving automorphism, then $L_2(M)$ can be equipped with a twisted representation $x \cdot \xi \cdot y = x\xi\tau(y)$.
- (iv) If $B \subseteq M$ is a von Neumann subalgebra, denote by $e_B: L_2(M) \rightarrow L_2(B)$ the orthogonal projection. The *basic construction*, $\langle M, e_B \rangle$ is the von Neumann subalgebra of $B(L_2(M))$ generated by M and e_B . Endowing $\langle M, e_B \rangle$ with the semifinite trace

$$\text{Tr}(xe_B y) = \tau(xy),$$

then $L_2(\langle M, e_B \rangle, \text{Tr})$ is endowed with the structure of a M - M -bimodule.

We may now consider a dictionary (of sorts) between bimodules and unital completely positive maps. We will then specialize to the case of group representations.

To start, let $\phi: M \rightarrow N$ be a trace-preserving unital completely positive map. Define a sesquilinear form on $M \otimes L_2(N)$ by

$$\left\langle \sum_i a_i \otimes \eta_i, \sum_j b_j \otimes \zeta_j \right\rangle = \sum_{i,j} \langle \phi(b_j^* a_i) \eta_i, \zeta_j \rangle_{L_2(N)},$$

and build a Hilbert space by completing with respect to this norm, which we denote $\mathcal{H}_\phi$. We may equip $\mathcal{H}_\phi$ with an M - N -bimodule structure by defining the action on $b \otimes \zeta$ by

$$a(b \otimes \zeta)x = (ab) \otimes (\zeta x)$$

for any $a \in M$ and $x \in N$.

Letting $\xi = 1 \otimes \hat{1}$ be the unit vector in $\mathcal{H}_\phi$, where $\hat{1}$ denotes the cyclic vector in $L_2(N)$, then we have

$$\langle a\xi x, \xi y \rangle_{\mathcal{H}_\phi} = \langle \phi(a)\hat{x}, \hat{y} \rangle_{L_2(N)}$$

for all $a \in M$ and $x, y \in N$. Since ϕ is assumed to be trace-preserving, we have that

$$\tau_m = \langle \cdot, \xi \rangle$$

$$\tau_N = \langle \xi, \cdot \rangle.$$

Now, let $\mathcal{H}$ be a M - N -bimodule with tracial unit vector ξ — i.e., $\tau_M = \langle \cdot, \xi \rangle$ and $\tau_N = \langle \xi, \cdot \rangle$. Then, the map $L_\xi: L_2(N) \rightarrow \mathcal{H}$ given by

$$L_\xi(\hat{x}) = \xi x$$

is bounded with operator norm 1. For any $x, y \in N$, we have

$$\begin{aligned} \langle \hat{x}, L_\xi^*(\xi y) \rangle &= \langle \xi x, \xi y \rangle \\ &= \tau_N(xy^*) \\ &= \langle \hat{x}, \hat{y} \rangle_{L_2(N)}. \end{aligned}$$

That is, $L_\xi^*(\xi y) = \hat{y}$. In particular, L_ξ is an isometry.

Denote by $J = J_N$ the canonical anti-linear unitary. For any $a \in M$, we have $L_\xi^* a L_\xi \in N$, since for any $y, z_1, z_2 \in N$, we have

$$\begin{aligned} \langle (L_\xi^* a L_\xi)(J y^* J) \hat{z}_1, \hat{z}_2 \rangle_{L_2(N)} &= \langle a L_\xi J y^* J \hat{z}_1, L_\xi \hat{z}_2 \rangle_{\mathcal{H}} \\ &= \langle a \xi z_1 y, \xi z_2 \rangle_{\mathcal{H}}. \end{aligned}$$

Meanwhile,

$$\begin{aligned} \langle (J y^* J)(L_\xi^* a L_\xi) \hat{z}_1, \hat{z}_2 \rangle_{L_2(N)} &= \langle a L_\xi \hat{z}_1, L_\xi J y^* J \hat{z}_2 \rangle_{\mathcal{H}} \\ &= \langle a \xi z_1, \xi z_2 y^* \rangle_{\mathcal{H}}. \end{aligned}$$

Therefore, we may define the unital completely positive map $\phi: M \rightarrow N$ by $\phi(a) = L_\xi^* a L_\xi$. We have that

$$\begin{aligned} \tau_N(\phi(a)) &= \langle L_\xi^* a L_\xi \hat{1}, \hat{1} \rangle \\ &= \langle a \xi, \xi \rangle \\ &= \tau_M(a), \end{aligned}$$

so that ϕ is trace-preserving. Furthermore,

$$\langle \phi(a) \hat{x}, \hat{y} \rangle = \langle a \xi x, \xi y \rangle.$$

This dictionary between unital completely positive maps and bimodules enables a different proof of the existence and uniqueness of a trace-preserving conditional expectation.

Proposition: Let (M, τ) be a finite von Neumann algebra, and let $B \subseteq M$ be a von Neumann subalgebra with $\tau_B = \tau|_B$. Then, there exists a unique normal faithful trace-preserving conditional expectation $E: M \rightarrow B$.

Proof. Equip $\mathcal{H} = L_2(M, \tau)$ with an M - B -bimodule structure. The vector $\hat{1} \in \mathcal{H}$ is a unit tracial vector. There is a corresponding normal trace-preserving unital completely positive map $E_B: M \rightarrow B$ from our work above. We claim this is our desired conditional expectation.

Recall that we have

$$\langle x \hat{1} a, \hat{1} b \rangle_{L_2(M)} = \langle E_B(x) \hat{a}, \hat{b} \rangle_{L_2(B)}$$

for all $x \in M$ and all $a, b \in B$. Then, for any $x \in M$ and any $a, b, c, d \in B$, we have

$$\begin{aligned} \langle E_B(axb) \hat{c}, \hat{d} \rangle_{L_2(B)} &= \langle axb \hat{1} c, \hat{1} d \rangle \\ &= \langle x \hat{1} bc, \hat{1} a^* d \rangle \\ &= \langle E_B(x) \widehat{bc}, \widehat{a^* d} \rangle \end{aligned}$$

$$= \langle aE_B(x)b\hat{c}, \hat{d} \rangle.$$

Therefore, E_B is a conditional expectation. If $E_B(x^*x) = 0$, then

$$\begin{aligned} 0 &= \langle E_B(x^*x)\hat{1}, \hat{1} \rangle \\ &= \langle x^*x\hat{1}, \hat{1} \rangle \\ &= \|x\|_\tau^2, \end{aligned}$$

meaning $x = 0$.

Finally, suppose E is another trace-preserving conditional expectation. Then,

$$\begin{aligned} \langle E_B(x)\hat{c}, \hat{d} \rangle &= \langle x\hat{1}c, \hat{1}d \rangle \\ &= \tau(d^*xc) \\ &= \tau(E(d^*xc)) \\ &= \tau(d^*E(x)c) \\ &= \langle E(x)\hat{c}, \hat{d} \rangle, \end{aligned}$$

so $E_B(x) = E(x)$, giving uniqueness. $\square$

Now, we can consider the translation between unitary representations of discrete groups and Hilbert bimodules. Let $\pi: \Gamma \rightarrow U(\mathcal{H}_\pi)$ be a unitary representation of a countable discrete group Γ , and let $M = L(\Gamma)$ be the group von Neumann algebra with canonical unitaries $(u_s)_{s \in \Gamma}$. Let $\mathcal{K}_\pi = \mathcal{H}_\pi \otimes \ell_2(\Gamma)$, and define left and right commuting multiplications by taking

$$\begin{aligned} u_s \cdot (\xi \otimes \delta_t) &= (\pi_s \otimes \lambda_s)(\xi \otimes \delta_t) \\ &= \pi_s \xi \otimes \delta_{st} \\ (\xi \otimes \delta_t) \cdot u_s &= (1 \otimes \rho_{s^{-1}})(\xi \otimes \delta_t) \\ &= \xi \otimes \delta_{ts}. \end{aligned}$$

We see that right multiplication extends to the whole von Neumann algebra, and that the unitary representations $\pi \otimes \lambda$ and $1 \otimes \lambda$ are unitarily equivalent by Fell's Absorption Principle. Therefore, left multiplication extends to M , so we have that $\mathcal{K}_\pi$ is equipped with the structure of an $L(\Gamma)$ - $L(\Gamma)$ -bimodule.

If π is the left regular representation of Γ , then the M - M -bimodule $\mathcal{K}_\lambda$ is the coarse bimodule $L_2(M) \otimes L_2(M)$.

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